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Higgs structures of dyonic instantons

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Abstract: We study Higgs field configurations of dyonic instantons in spontaneously broken $(4+1)$-dimensional Yang-Mills theory. The adjoint scalar field solutions to the covariant Laplace equation in the ADHM instanton background are constructed in general noncanonical basis, and they are used to study explicitly the Higgs field configurations of dyonic instantons when the gauge fields are taken by Jackiw-Nohl-Rebbi instanton solutions. For these solutions corresponding to small instanton number we then consider in some detail the zero locus of the Higgs field, which describes the cross section of supertubes connecting parallel D4-branes in string theory. Also the information on the Higgs zeroes is used to discuss the residual gauge freedom concerning the Jackiw-Nohl-Rebbi solutions.

Keywords: Solitons Monopoles and Instantons, D-branes, Supersymmetric gauge theory.

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Contents

1. Introduction

Instanton solutions of 4-dimensional Euclidean Yang-Mills theory are also known to play a role as solitons in certain spontaneously broken $(4+1)$-dimensional gauge theory. Corresponding objects, first discussed by Lambert and Tong [1], are called dyonic instantons, for stable instantons in the broken vacuum must come with nonzero electric charge. In type IIA string theory the D-brane interpretation of dyonic instantons can be found in a supertube [2−4] which connects parallel D4-branes lying close to each other. See also refs. [5−9] for discussions relevant to dyonic instantons from the latter perspective.

Classical dyonic instanton solutions, as described by Yang-Mills gauge field A_μ and Higgs scalar ϕ (which are functions of four spatial coordinates x_μ), satisfy Bogomol'nyi-type equations: especially, $A_\mu(x)$ satisfy the usual self-duality conditions appropriate to Yang-Mills instantons

$$F_{\mu\nu} = {}^*F_{\mu\nu} \tag{1.1}$$

while the scalar fields (in the adjoint representation) $\phi(x)$ obey the covariant Laplace equation in the background of the instanton

$$D_\mu D_\mu \phi = 0\ . \tag{1.2}$$

The gauge fields can thus be described by the ADHM construction [11−13], which includes the Jackiw-Nohl-Rebbi (JNR) instanton solutions [14] as (explicit) special cases. The adjoint Higgs solutions to (1.2) are easy to find when the gauge fields are taken by 't Hooft solutions [15], but, with the JNR instanton backgrounds, complete expressions are known only for the two-instanton case [7]. The latter expressions were found using the construction of refs. [16 − 18], where the appropriate form for the Higgs field was identified when the ADHM data for the background gauge fields was presented in the canonical basis. It is much desirable to have more systematic understanding on the solutions to (1.2) in non-'t Hooft-type instanton backgrounds since the zero locus of these Higgs fields is known to have direct connection with the cross section of the (noncollapsed) supertubes.

In the present work we have two goals. The first is to provide a fuller analysis on the solutions of (1.2), in their general structure and also by obtaining new explicit solutions in JNR backgrounds. The second is to use such knowledge on the Higgs configurations, especially their zero locus, to clarify further various issues related to the supertube interpretation of dyonic instantons. In this work we find Higgs fields satisfying the covariant Laplace equation in general ADHM backgrounds (given in arbitrary basis, canonical or not), directly from the asymptotic behavior of the known scalar propagator [12, 19] in ADHM backgrounds. It is seen that the complex structure Higgs solutions have is entirely due to the same matrix factor that also enters the propagator expression for adjoint scalar. To find explicit Higgs solutions in JNR backgrounds, we can now utilize our construction directly (without going through the awkward procedure [7] as needed to express JNR instanton solutions using canonical ADHM data first); fully explicit forms of the Higgs solutions can be produced this way. Then, as in ref. [7], the zero locus of the Higgs fields can be studied. On the special significance carried by these Higgs zeroes, some explanations will be offered below.

As noted in ref. [7], the zero locus of the Higgs field is the magnetic monopole string along which the unbroken U(1) magnetic flux emerges. A loop-like boundary of a tubular D2-brane connecting two parallel but separated D4-branes would appear as the magnetic string on D4 worldvolume. Let us recall that a supertube is made of a tube of D2-brane with fundamental strings (F1) lying along the tube direction and D0-branes spread along the D2-brane such that the F1 string number times the D0-brane number density remains constant [3]. Also the straight D2-brane connecting two parallel but separated D4-branes appears as a U(1) magnetic monopole string of (4+1)-dimensional Yang-Mills theory. Thus the zero locus is the direct indicator of the way a supertube connects two D4-branes, and on this aspect we shall expand the discussions of ref. [7] further by studying the zeroes of the explicit Higgs solutions obtained in this work. A supertube with large D0 and F1 charges can have arbitrary cross section. Similarly, the number of moduli parameters for the magnetic monopole string would increase with the instanton number. If κ is the instanton number, then the dimension of instanton moduli space is 8κ. In the Coulomb phase, the fixed electric charge gives one constraint and the corresponding coordinate is cyclic. Thus dyonic instantons of the instanton number κ in the center of mass frame has $8\kappa - 6$ independent parameters which also characterize the shape of the magnetic monopole string.

Explicit instanton solutions/Higgs configurations are hard to obtain in general so that the structure of the zero locus as the representation of the cross section of supertubes is difficult to get in full detail. For the JNR-type dyonic instanton with the instanton number κ, the number of relevant parameters is given by $5\kappa + 6$ or $5\kappa + 7$, depending on whether the JNR position parameters are aligned along a circle (or a straight line) or not. While the number of parameters in the JNR-type solutions is less than 8κ (as appropriate for the most general SU(2) dyonic instanton solutions), the Higgs field of the JNR-type solutions have an intricate structure to warrant further investigations. In this work, we find explicit Higgs solutions for the JNR-type dyonic instanton with the instanton numbers $\kappa = 3$ and $\kappa = 4$ and obtain explicit expressions for the electric charge with some small value of κ. We then elaborate on the structures of the zero locus of the thus-obtained Higgs field, expanding the similar analysis done in ref. [7] significantly. The global gauge orientation of the instanton configurations with respect to the asymptotic Higgs expectation value plays an important role in changing the shape of the zero locus. Also analyzed is the reduction of the number of moduli which happens when all the JNR position parameters are aligned on a circle [14], by utilizing the gauge invariance of the zero locus of the Higgs field.

This paper is organized as follows. In section 2 we identify the Higgs field forms of dyonic instantons, that go with ADHM instanton fields in general basis. (We here assume spontaneously broken SU(2) gauge theory). The electric charge is also computed for general dyonic instantons. Section 3 is devoted to finding the explicit Higgs solutions in JNR backgrounds, and we see some regular patterns emerging. Based on the findings of section 3, we then analyze in section 4 the Higgs zero locus structure in some detail and examine the related D4-brane-supertube configurations. We also use the information on the Higgs zeroes to study the residual gauge freedom that enters the general JNR instanton solutions. Section 5 contains conclusion and discussions for future study. There are two appendices. In appendix A we provide a direct verification for our Higgs field construction in general ADHM backgrounds. Appendix B contains certain technical parts relevant in the computation of electric charge.

2. The ADHM construction of dyonic instantons in general basis

We will start with presenting the Bogomol'nyi-type equations describing dyonic instantons in $(4+1)$-dimensional Yang-Mills-Higgs theory [1]. We here have the energy functional

$$\mathcal{E} = -\frac{1}{e^2}\int d^4x \ \mathrm{tr}\left\{E_\mu^2 + \frac{1}{2}F_{\mu\nu}^2 + (D_t\phi)^2 + (D_\mu\phi)^2\right\}, \tag{2.1}$$

where $\mu = 0,1,2,3$ is a spatial index, $D_\mu \equiv \partial_\mu + [A_\mu, \ ]$ (D_t likewise), $E_\mu \equiv F_{\mu t} = D_\mu A_t - \partial_t A_\mu$, and ϕ denote adjoint scalars. Both (A_t, A_μ) and ϕ are taken to be antihermitian matrices in the space of gauge group generators. Then, completing the square in the usual fashion and using the Gauss law

$$D_\mu E_\mu - [\phi, D_t\phi] = 0\,, \tag{2.2}$$

it is possible to rearrange this energy functional into the form

$$\mathcal{E} = -\frac{1}{e^2}\int d^4x \ \mathrm{tr}\left\{(E_\mu - D_\mu\phi)^2 + \frac{1}{4}(F_{\mu\nu} - {}^*F_{\mu\nu})^2 + (D_t\phi)^2\right\} + \frac{8\pi^2}{e^2}\kappa + \frac{2\mathrm{v}}{e^2}Q_e\,, \tag{2.3}$$

where the quantities

$$\kappa = -\frac{1}{16\pi^2}\int d^4x \ \mathrm{tr}(F_{\mu\nu}{}^*F_{\mu\nu}) \tag{2.4}$$

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and

$$Q_e = -\int d^4x \; \partial_\mu \frac{1}{\mathrm{v}} \mathrm{tr}(\phi E_\mu) \tag{2.5}$$

denote the instanton number and the electric charge, respectively. In (2.5) v denotes a constant Higgs vacuum expectation value, with $\mathrm{v}^2 = \lim_{|x|\to\infty}(-2\mathrm{tr}\phi^2)$. When the values of $\kappa(\geq 0)$ and $Q_e(\geq 0)$ are given, we obtain the energy bound $\mathcal{E} \geq \left(\frac{8\pi^2}{e^2}\kappa + \frac{2\mathrm{v}}{e^2}Q_e\right)$ from (2.3). Further, this energy bound is saturated by the static field configurations satisfying

$$F_{\mu\nu} = {}^*F_{\mu\nu}\,, \quad E_\mu = D_\mu\phi\,, \quad D_t\phi = 0\,, \tag{2.6}$$

which are BPS equations for dyonic instantons. We can choose a gauge where $A_t = \phi$ in which case the fields are static in time.

From the first equation of (2.6), the (spatial) gauge fields of dyonic instantons are taken by usual Yang-Mills instanton solutions. On the other hand, since the electric field E_μ of the field configurations should satisfy the Gauss law constraint (2.2), the last two equations of (2.6) can be combined with the Gauss law to give

$$D_\mu D_\mu \phi = 0\ . \tag{2.7}$$

Hence the adjoint scalar fields of dyonic instantons should satisfy the covariant Laplace equation in the background of instanton solutions (and, at spatial infinity, approach the suitably chosen vacuum expectation values). Any nontrivial solution to (2.7) should give rise to a nonvanishing electric charge contribution to the energy, i.e., $Q_e \neq 0$. Authors of refs. [1, 7] obtained explicit dyonic instanton solutions for some special cases.

The general solution to Yang-Mills self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ is given by the ADHM construction [11−13]. Assuming SU(2) gauge group the ADHM form for gauge fields $A_\mu(x) \equiv A^a_\mu(x)\frac{\sigma_a}{2i}$ (σ_a, $a = 1, 2, 3$ denote three 2×2 Pauli matrices) corresponding to κ instantons is

$$A_\mu(x) = \sum_{l=0}^{\kappa} \bar{v}_l(x)\partial_\mu v_l(x) \equiv v^\dagger(x)\partial_\mu v(x)\,, \tag{2.8}$$

where $v_l(x) \equiv v_l^\mu(x)e_\mu$, $\bar{v}_l(x) \equiv v_l^\mu(x)\bar{e}_\mu$ with $e_\mu \equiv (e_0 = 1, \vec{e} = -i\vec{\sigma})$ and $\bar{e}_\mu \equiv (1, i\vec{\sigma})$ (i.e., $v_l(x)$ and $\bar{v}_l(x)$ are quaternionic objects), and $v(x)$ a quaternionic $(\kappa+1)$-column vector. Then, because of the self-duality equations, the quaternionic column vector $v(x)$ is constrained by the equations

$$\begin{aligned} v^\dagger(x)v(x) &= 1\,, \\ \Delta^\dagger(x)v(x) &= v^\dagger(x)\Delta(x) = 0\,, \end{aligned} \tag{2.9}$$

where Δ, a quaternionic $(\kappa+1)\times\kappa$ matrix with linear x-dependence

$$\Delta = B - Cx\,, \quad (x \equiv x^\mu e_\mu) \tag{2.10}$$

should obey the so-called ADHM constraint: the $\kappa\times\kappa$ matrix $\Delta^\dagger(x)\Delta(x)$ must be real (i.e., belong to the identity element of quaternion) and invertible. The nontrivial part of this method is to find two constant quaternionic $(\kappa+1)\times\kappa$ matrices B and C, and for $\kappa > 3$ the corresponding expressions in their full generality are still not known.

Now our task is to determine the broken-vacuum solution $\phi(x) \equiv \phi^a(x)\frac{\sigma_a}{2i}$ to (2.7) with the above ADHM background for A_μ. By the broken vacuum we mean that when $A_\mu(x)$ at spatial infinity is described by a pure gauge of the form

$$A_\mu \to \bar{g}(\hat{x})\partial_\mu g(\hat{x}) \ , \quad \text{as } |x| \to \infty \tag{2.11}$$

(here $\hat{x}^\mu \equiv x^\mu/|x|$ and $g(\hat{x})$ is an arbitrary unit quaternion), the asymptotic behavior of our Higgs solution is

$$|x| \to \infty \ : \quad \phi(x) \to \bar{g}(\hat{x})\phi_0 g(\hat{x}) \tag{2.12}$$

with a *constant* matrix ϕ_0 ($|\phi_0| \equiv \sqrt{\phi_0^a\phi_0^a} = \mathrm{v}$); ϕ_0 may be identified with the scalar vacuum expectation value in the gauge where $A_\mu(x) < \mathcal{O}(1/|x|)$ as $|x| \to \infty$. For $\phi(x)$ belonging to the fundamental representation of SU(2), this task of solving the covariant Laplace equation would have been a easy one; the solution with a very simple structure (see below) was obtained already in ref. [20]. But, in our case, i.e., with an adjoint scalar, the solution is not so simple due to certain complications known to occur when fields in higher dimensional representations are involved. A systematic way to deal with the problem is the tensor product method of Corrigan et al [19]. This method was used by them to understand the structure of adjoint scalar propagators [21, 12] and also to obtain the (adjoint representation) solution to massless Dirac equation in the ADHM background.

Actually, for the solution of (2.7), it is unnecessary to go through the tensor product formalism of ref. [19] — we can instead exploit the known expression for the scalar propagator in the ADHM instanton background. For the sake of comparison, the solution to (2.7) for both fundamental and adjoint scalar ϕ will be considered. For the fundamental scalar, the propagator or the inverse covariant Laplacian is [12, 13]

$$\Delta^{(\frac{1}{2})}(x,y) = \frac{v^\dagger(x)v(y)}{4\pi^2(x-y)^2} \,, \tag{2.13}$$

using 2×2 matrix notations. On the other hand, the corresponding expression for the adjoint scalar has more complicated structure [12, 19]

$$\begin{aligned}\Delta^{(1)}_{ab}(x,y) = {} & \frac{1}{8\pi^2(x-y)^2}\,\mathrm{tr}\left\{\sigma_a v^\dagger(x)v(y)\sigma_b v^\dagger(y)v(x)\right\} \\ & + \frac{1}{8\pi^2}\sum_{ijmn=1}^{\kappa}\mathrm{tr}\left\{C^\dagger v(x)\sigma_a v^\dagger(x)C\right\}_{ij} f_{ij,mn}\mathrm{tr}\left\{C^\dagger v(y)\sigma_b v^\dagger(y)C\right\}_{mn} \,,\end{aligned} \tag{2.14}$$

where 'tr' refers to the trace of 2×2 matrix representing quaternionic quantities, and the constants $f_{ij,mn}$ are specified through the matrix equation

$$f_{ij,mn}L_{mn,rs} = \delta_{ir}\delta_{js} - \delta_{jr}\delta_{is} \,, \tag{2.15}$$

$$\begin{aligned}L_{mn,rs} = {} & \frac{1}{2}\mathrm{tr}\left\{2\left(C^\dagger B\right)_{mr}\left(B^\dagger C\right)_{sn} - \left(C^\dagger C\right)_{mr}\left(B^\dagger B\right)_{sn} - \left(B^\dagger B\right)_{mr}\left(C^\dagger C\right)_{sn}\right\} \\ & -(m \leftrightarrow n) \ .\end{aligned} \tag{2.16}$$

(It was to understand the necessity of the second term on the right of (2.14), as first noticed in ref. [21], that the tensor product method was originally developed [19]). Now, taking the asymptotic limit $|y| \to \infty$ with these propagators, we may express the results by

$$|y| \to \infty \quad : \quad \Delta^{(\frac{1}{2})}(x,y) = \frac{\Phi^{(\frac{1}{2})}(x,\hat{y})}{|y|^2} + \mathcal{O}\left(\frac{1}{|y|^3}\right) , \tag{2.17a}$$

$$\Delta^{(1)}_{ab}(x,y) = \frac{\Phi^{(1)}_{ab}(x,\hat{y})}{|y|^2} + \mathcal{O}\left(\frac{1}{|y|^3}\right) . \tag{2.17b}$$

Then, from the defining equations satisfied by the propagators, the functions $\Phi^{(\frac{1}{2})}(x,\hat{y})$ and $\Phi^{(1)}_{ab}(x,\hat{y})$ will have to satisfy the appropriate covariant Laplace equations, i.e.,

$$D^{(x)}_\mu D^{(x)}_\mu \Phi^{(\frac{1}{2})}(x,\hat{y}) = 0 , \tag{2.18a}$$

$$(D^{(x)}_\mu D^{(x)}_\mu)_{ab} \Phi^{(1)}_{bc}(x,\hat{y}) = 0 , \tag{2.18b}$$

the unit vector $\hat{y}$ serving only as free parameters. Based on this, we can identify the solution to (2.7) (up to a multiplicative constant) with

$$\phi^{(\frac{1}{2})}_\alpha(x) = \Phi^{(\frac{1}{2})}_{\alpha\beta}(x,\hat{y}) u_\beta , \quad \text{(fundamental)} \tag{2.19a}$$

$$\phi^{(1)}_a(x) = \Phi^{(1)}_{ab}(x,\hat{y}) u_b , \quad \text{(adjoint)} \tag{2.19b}$$

where u_β ($\beta = 1,2$), u_b ($b = 1,2,3$) denote arbitrary constant isospinor and isovector, respectively. This is our key observation.

To find Higgs solutions by the above method, we need the asymptotic behavior of a quaternionic $(\kappa+1)$-column vector $v(y)$. Because of the first condition in (2.9), we may here write

$$v(y) = h_0(\hat{y}) + h_1(\hat{y})\frac{1}{|y|} + \mathcal{O}\left(\frac{1}{|y|^2}\right) , \quad \text{as } |y| \to \infty . \tag{2.20}$$

Then, from (2.8), (2.11) and the two conditions in (2.9), we are led to following conclusions:

(i) $h_0(\hat{y})$ can be expressed as

$$h_0(\hat{y}) = V g(\hat{y}) \quad (\text{or } h_0(\hat{y})_l = V_l\, g(\hat{y}) , \ l = 0,1,\ldots,\kappa) \tag{2.21}$$

with a constant quaternionic $(\kappa+1)$-column vector V satisfying the conditions

$$V^\dagger V = 1 , \quad C^\dagger V = 0 . \tag{2.22}$$

(ii) $h_1(\hat{y})$ should satisfy the condition $C^\dagger h_1(\hat{y}) = \hat{y} B^\dagger h_0(\hat{y})$, or using (2.21),

$$C^\dagger h_1(\hat{y}) = \hat{y} B^\dagger V g(\hat{y}) . \tag{2.23}$$

We then notice from (2.13) and (2.17a) that $\Phi^{(\frac{1}{2})}(x,\hat{y}) \propto v^\dagger(x) h_0(\hat{y}) = v^\dagger(x) V g(\hat{y})$ and therefore the fundamental scalar solution of (2.7) is simply

$$\phi^{(\frac{1}{2})}(x) = v^\dagger(x) V g(\hat{y}) u = v^\dagger V \phi_0 , \tag{2.24}$$

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where we replaced $g(\hat{y})u$ by the appropriate vacuum expectation value ϕ_0 (in accordance with the asymptotic requirement $\phi(x) \to \bar{g}(\hat{x})\phi_0$ as $|x| \to \infty$). Our expression (2.24) agrees with the result obtained in ref. [20].

For the function $\Phi^{(1)}_{ab}(x, \hat{y})$ appropriate to an adjoint scalar, it is not difficult to see that the first piece in the corresponding scalar propagator (2.14) contributes a term proportional to $\mathrm{tr}\{\sigma_a v^\dagger(x) V g(\hat{y})\sigma_b \bar{g}(\hat{y}) V^\dagger v(x)\}$. The contribution from the second piece can also be found easily if one uses the observation (resulting from (2.20)–(2.23))

$$C^\dagger v(y) = \frac{1}{|y|}\hat{y} B^\dagger V g(\hat{y}) + \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad \text{as } |y| \to \infty \tag{2.25}$$

and its consequence

$$\mathrm{tr}\left\{C^\dagger v(y)\sigma_b v^\dagger(y) C\right\}_{mn} = \frac{1}{|y|^2}\mathrm{tr}\left\{B^\dagger V g(\hat{y})\sigma_b \bar{g}(\hat{y}) V^\dagger B\right\}_{mn} + \mathcal{O}\left(\frac{1}{|y|^3}\right). \tag{2.26}$$

(Note that $\hat{y}\hat{\bar{y}} = 1$). In this way, for the solution of (2.7), we obtain the expression

$$\begin{aligned} \phi(x) \equiv \phi^{(1)}_a(x)\frac{\sigma_a}{2i} &= \frac{\sigma_a}{2i}\Phi^{(1)}_{ab}(x, \hat{y}) u_b \\ &= \frac{1}{2}\sum_a \sigma_a \mathrm{tr}\left\{\sigma_a v^\dagger(x) V \phi_0 V^\dagger v(x)\right\} + \sum_{ij=1}^{\kappa}\sum_a \sigma_a \mathrm{tr}\left\{C^\dagger v(x)\sigma_a v^\dagger(x) C\right\}_{ij}\mathcal{A}_{ij}, \end{aligned} \tag{2.27}$$

where we have made the identification (see below)

$$g(\hat{y})\vec{u}\cdot\frac{\vec{\sigma}}{2i}\bar{g}(\hat{y}) = \phi_0, \tag{2.28}$$

and introduced the antisymmetric matrix $\mathcal{A} = (\mathcal{A}_{ij})$ determined by solving the inhomogeneous linear simultaneous equations

$$L_{mn,rs}\mathcal{A}_{rs} = \mathrm{tr}(B^\dagger V \phi_0 V^\dagger B)_{mn}. \tag{2.29}$$

The identification (2.28) is the result of comparing the asymptotic behavior of our expression against (2.12). Here, the second term in (2.27) being $\mathcal{O}(1/|x|^2)$ as $|x| \to \infty$, it suffices to consider the first term which indeed reduce to $\frac{1}{2}\sum_a \sigma_a \mathrm{tr}\{\sigma_a \bar{g}(\hat{x}) V^\dagger V \phi_0 V^\dagger V g(\hat{x})\} = \bar{g}(\hat{x})\phi_0 g(\hat{x})$ as $|x| \to \infty$. We further note that, since $\frac{1}{2}\sum_a \sigma_a \mathrm{tr}\{\sigma_a X\} = X$ if X is traceless, (2.27) can be simplified into the form

$$\phi(x) = v^\dagger(x) V \phi_0 V^\dagger v(x) - \sum_{ij=1}^{\kappa}\left[\left(v^\dagger(x) C\right)_i \left(C^\dagger v(x)\right)_j - \left(v^\dagger(x) C\right)_j \left(C^\dagger v(x)\right)_i\right]\mathcal{A}_{ij}. \tag{2.30}$$

This is the expression we have been after. We also verified explicitly that our expression (2.30) solves (2.7); this direct check is not quite trivial (like many other calculations involving instanton solutions) and so, for interested readers, we provide some essential steps needed in the verification in appendix A.

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With the *canonical* ADHM data assumed, i.e.,when the quaternionic $(\kappa+1)\times\kappa$ matrix Δ is presented in the form

$$\Delta = \begin{pmatrix} \Lambda^\mu e_\mu \\ \hline \\ \Omega^\mu e_\mu \\ \\ \end{pmatrix} - \begin{pmatrix} 0 \\ \hline \\ I_{\kappa\times\kappa} \\ \\ \end{pmatrix} x \equiv B - Cx \tag{2.31}$$

($\Lambda^\mu = (\Lambda^\mu_1 \cdots \Lambda^\mu_\kappa)$ is a κ-row vector, and $\Omega^\mu = (\Omega^\mu_{mn})$ a $\kappa \times \kappa$ hermitian matrix), (2.30) reduces to the result of refs. [16 – 18]. This can be seen as follows. In this canonical basis, the above quaternionic $(\kappa+1)$ column vector V becomes simply

$$V = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.32}$$

and as a result our expression (2.30) can be organized into

$$\phi(x) = \bar{v}_0(x)\phi_0 v_0(x) - 2\sum_{ij=1}^{\kappa} \bar{v}_i(x)\mathcal{A}_{ij}v_j(x) = v^\dagger(x) \left(\begin{array}{c|c} \phi_0 & 0 \\ \hline & \\ 0 & -2\mathcal{A} \\ & \end{array}\right) v(x)\ . \tag{2.33}$$

At the same time, the linear equations (2.29) for the matrix $\mathcal{A}$ (with $L_{mn,rs}$ given in (2.16)) can be simplified using following results valid in this basis:

$$\begin{aligned} &C^\dagger B = \Omega (= \Omega^\mu e_\mu)\,, && B^\dagger C = \Omega^\dagger (= \Omega^\mu \bar{e}_\mu)\,, && B^\dagger B = \Lambda^\dagger \Lambda + \Omega^\dagger \Omega\,, \\ &C^\dagger C = I\,, && B^\dagger V = \Lambda^\dagger\,, && V^\dagger B = \Lambda\,, \end{aligned} \tag{2.34}$$

and, for the inhomogeneous term in (2.29),

$$\mathrm{tr}(B^\dagger V \phi_0 V^\dagger B)_{mn} = -2i\eta_{\mu\nu a}\phi_0^a \left[\left(\Lambda^\dagger\right)^\mu \Lambda^\nu\right]_{mn}, \tag{2.35}$$

where we used the identities like $\sigma_a e_\nu = i\eta_{\nu\lambda a}e_\lambda$ and $\bar{e}_\mu e_\nu = \delta_{\mu\nu} + i\bar{\eta}_{\mu\nu a}\sigma_a$. (Note that, in our notation, $e_\mu \bar{e}_\nu = \delta_{\mu\nu} + i\eta_{\mu\nu a}\sigma_a$). Then one finds that the matrix $\mathcal{A}$ should satisfy the equation

$$-[\Omega^\mu, [\Omega^\mu, \mathcal{A}]] - \left\{\left(\Lambda^\dagger\right)^\mu \Lambda^\mu, \mathcal{A}\right\} = -i\eta_{\mu\nu a}\phi_0^a \left(\Lambda^\dagger\right)^\mu \Lambda^\nu\,, \tag{2.36}$$

which is precisely what the authors of refs. [16 – 18] obtained as the condition for $\mathcal{A}$.

The 't Hooft κ instanton solution [15]

$$A_\mu(x) = -\bar{\eta}_{\mu\nu a}\frac{\sigma_a}{2i}\partial_\nu \log\tilde{\Pi}(x)\,, \quad \tilde{\Pi}(x) = 1 + \sum_{m=1}^{\kappa}\frac{\rho_m^2}{|x - z_m|^2} \tag{2.37}$$

can be obtained from the simple canonical ADHM data: explicitly, we here have ($z_m \equiv z_m^\mu e_\mu$)

$$B = \begin{pmatrix} \rho_1 & \cdots & \rho_\kappa \\ \hline z_1 & & 0 \\ & \ddots & \\ 0 & & z_\kappa \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad v(x) = \frac{1}{\sqrt{\tilde{\Pi}(x)}} \begin{pmatrix} 1 \\ \frac{\rho_1(x-z_1)}{|x-z_1|^2} \\ \vdots \\ \frac{\rho_\kappa(x-z_\kappa)}{|x-z_\kappa|^2} \end{pmatrix}. \tag{2.38}$$

In this case, a simple calculation shows that the inhomogeneous term (2.35) vanishes and hence $\mathcal{A} \equiv 0$. The Higgs solution to the covariant Laplace equation is thus given by [7]

$$\phi(x) = \bar{v}_0(x)\phi_0 v_0(x) = \frac{1}{\tilde{\Pi}(x)}\phi_0 . \tag{2.39}$$

But, with the JNR κ instanton solution (which is described more simply using noncanonical ADHM data), one finds $\mathcal{A} \neq 0$ if $\kappa > 1$ and cannot expect this sort of simple solutions any longer. For this JNR case, see section 3 for detailed discussions.

General SU(2) dyonic instantons can be described by the ADHM construction for $A_\mu(x)$ and the corresponding Higgs solution (2.30). Using these solutions, we can evaluate the electric charge Q_e (given by (2.5)) explicitly. Relegating the details of calculation to appendix B, we here give the final result only:

$$Q_e = -\frac{4\pi^2}{\mathrm{v}}\mathrm{tr}\left\{\phi_0^2 V^\dagger B \left(C^\dagger C\right)^{-1} B^\dagger V + 2\phi_0 V^\dagger B \mathcal{A} B^\dagger V\right\} . \tag{2.40}$$

(Here we invoked matrix notation in which (2.30) can be written as $\phi = v^\dagger V \phi_0 V^\dagger v - 2v^\dagger C \mathcal{A} C^\dagger v$). If one takes the ADHM solution given in the canonical basis and the corresponding Higgs solution (2.33) (the relevant data are given in (2.34)), (2.40) reduces to the simple form

$$Q_e = 2\pi^2(\Lambda_i^\mu)\left[\mathrm{v}\delta_{\mu\nu}\delta_{ij} - \frac{4\eta_{\mu\nu a}\phi_0^a}{\mathrm{v}}\mathcal{A}_{ij}\right](\bar{\Lambda}_j^\nu) \tag{2.41}$$

with the matrix $\mathcal{A}$ obtained by solving (2.36). An equivalent expression to this result was found in ref. [1]. For dyonic instantons given by the 't Hooft configuration (2.37) and the corresponding Higgs field (2.39), we find from (2.40) or (2.41) the value $Q_e = 2\pi^2\mathrm{v}(\rho_1^2 + \cdots + \rho_\kappa^2)$, which is positive as $\mathrm{v} > 0$. The above electric charge should also be positive.

3. Higgs configurations with the Jackiw-Nohl-Rebbi instantons

The Higgs solution (2.39), obtained in the 't Hooft instanton background, has κ isolated zeroes at instanton positions $x = z_1, \ldots, z_\kappa$; as argued in ref. [7], these solutions describe collapsed supertubes connecting D4-branes. To see any indication of supertubes with finite size which connect two D4-branes, it was suggested [7] to study the Higgs configuration in the JNR background [14]. The latter is obtained by modifying (2.37) into the form

$$A_\mu(x) = -\bar{\eta}_{\mu\nu a}\frac{\sigma_a}{2i}\partial_\nu \log \Pi(x), \quad \Pi(x) = \sum_{l=0}^{\kappa} \frac{\rho_l^2}{|x-z_l|^2} . \tag{3.1}$$

This κ instanton configuration, with four more physically relevant parameters than the 't Hooft instanton configuration (2.37),[1] can be obtained from the noncanonical ADHM data [13, 7]

$$B = \begin{pmatrix} -\frac{\rho_1}{\rho_0}z_0 & -\frac{\rho_2}{\rho_0}z_0 & \cdots & -\frac{\rho_\kappa}{\rho_0}z_0 \\ \hline z_1 & & & 0 \\ & z_2 & & \\ & & \ddots & \\ 0 & & & z_\kappa \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{\rho_1}{\rho_0} & -\frac{\rho_2}{\rho_0} & \cdots & -\frac{\rho_\kappa}{\rho_0} \\ \hline 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \tag{3.2}$$

With these data the quaternionic column vector $v(x)$ is readily found to be

$$v(x) = \frac{1}{\sqrt{\Pi(x)}} \begin{pmatrix} \frac{\rho_0(x-z_0)}{|x-z_0|^2} \\ \frac{\rho_1(x-z_1)}{|x-z_1|^2} \\ \vdots \\ \frac{\rho_\kappa(x-z_\kappa)}{|x-z_\kappa|^2} \end{pmatrix}, \tag{3.3}$$

and hence, for the quantity $h_0(\hat{y}) = Vg(\hat{y})$ (see (2.21)), we have

$$g(\hat{y}) = \hat{y}\,, \quad V = \frac{1}{\sqrt{S}} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_\kappa \end{pmatrix}, \quad \left(S \equiv \sum_{l=0}^{\kappa} \rho_l^2 \right). \tag{3.4}$$

The Higgs solution in the JNR instanton background can be calculated by using the above data with our formula (2.30). The first contribution in (2.30), denoted $\phi_{\mathrm{I}}(x)$, is simple:

$$\begin{aligned} \phi_{\mathrm{I}}(x) &\equiv v^\dagger(x) V \phi_0 V^\dagger v(x) \\ &= \frac{1}{S\,\Pi(x)} \bar{\mathcal{F}}(x) \phi_0 \mathcal{F}(x)\,, \quad \left(\mathcal{F}(x) \equiv \sum_{l=0}^{\kappa} \frac{\rho_l^2 (x-z_l)}{|x-z_l|^2} \right) \end{aligned} \tag{3.5}$$

where S is a constant factor defined in (3.4). More explicitly, we can write this as

$$\phi_{\mathrm{I}}(x) = -\frac{1}{S\,\Pi(x)} \frac{\sigma_a}{2i} \left[\sum_{ll'=0}^{\kappa} \rho_l^2 \rho_{l'}^2 \, X_l^\mu X_{l'}^\nu \, \bar{\eta}_{\mu\lambda a} \eta_{\nu\lambda b} \right] \phi_0^b\,, \tag{3.6}$$

where $X_l^\mu(x) \equiv (x-z_l)^\mu / |x-z_l|^2$ $(l = 0, 1, \ldots, \kappa)$. In the present case the second term in (2.30) also contributes to the Higgs configuration if $\kappa > 1$, since we now find, for the inhomogeneous term in (2.29),

$$\mathrm{tr}\left(B^\dagger V \phi_0 V^\dagger B \right)_{mn} = -\frac{1}{S} \rho_m \rho_n \eta_{\lambda\delta b} \phi_0^b (z_m - z_0)^\lambda (z_n - z_0)^\delta \tag{3.7}$$

[1] When the $\kappa + 1$ points $z_0, z_1, \ldots, z_\kappa$ lie on a circle or a line, there is a residual gauge degree of freedom to reduce the number of parameters to $5\kappa + 3$, three more than the 't Hooft case [14].

and so some $\mathcal{A}_{ij}$ should not be zero. By substituting the ADHM data (3.2) in (2.16), we here obtain

$$\begin{aligned} L_{mn,rs} &= \left[-(z_m-z_n)^2\delta_{mr}\delta_{ns} - \frac{\rho_n\rho_s}{\rho_0^2}(z_m-z_0)^2\delta_{mr} - \frac{\rho_m\rho_r}{\rho_0^2}(z_n-z_0)^2\delta_{ns}\right] - [m\leftrightarrow n] \\ &= -\rho_m\rho_n\rho_r\rho_s\left[\frac{(z_m-z_n)^2}{\rho_m^2\rho_n^2}(\delta_{mr}\delta_{ns}-\delta_{ms}\delta_{nr}) + \frac{(z_m-z_0)^2}{\rho_0^2\rho_m^2}(\delta_{mr}-\delta_{ms})\right. \\ &\qquad\qquad \left. + \frac{(z_n-z_0)^2}{\rho_0^2\rho_n^2}(\delta_{ns}-\delta_{nr})\right] . \end{aligned} \tag{3.8}$$

Now, based on the expressions (3.7) and (3.8), our linear simultaneous equations (2.29) for the present case can be recast into the form

$$\left[\tilde{z}_{mn}^2(\delta_{mr}\delta_{ns}-\delta_{ms}\delta_{nr}) + \tilde{z}_{m0}^2(\delta_{mr}-\delta_{ms}) + \tilde{z}_{n0}^2(\delta_{ns}-\delta_{nr})\right]\tilde{\mathcal{A}}_{rs} = \frac{1}{S}\eta_{\lambda\delta b}\phi_0^b\left(z_0^\lambda z_m^\delta + z_m^\lambda z_n^\delta + z_n^\lambda z_0^\delta\right), \tag{3.9}$$

where we defined

$$\tilde{\mathcal{A}}_{rs} \equiv \rho_r\rho_s\mathcal{A}_{rs}\,, \quad \tilde{z}_{kl}^2 \equiv \frac{(z_k-z_l)^2}{\rho_k^2\rho_l^2}\,, \quad (k,l=0,1,\ldots,\kappa)\,. \tag{3.10}$$

From the very structure of the linear equations (3.9), we will also define the corresponding inverse matrix $\tilde{f}_{rs,mn}$ so that we may write

$$\tilde{\mathcal{A}}_{rs} = -\frac{1}{2S}\eta_{\lambda\delta b}\phi_0^b\sum_{mn=1}^{\kappa}\tilde{f}_{rs,mn}\left(z_0^\lambda z_m^\delta + z_m^\lambda z_n^\delta + z_n^\lambda z_0^\delta\right)\,. \tag{3.11}$$

At the same time, since we find from the expressions in (3.2) and (3.3)

$$\left(v^\dagger(x)C\right)_i\left(C^\dagger v(x)\right)_j - \left(v^\dagger(x)C\right)_j\left(C^\dagger v(x)\right)_i = \frac{\rho_i\rho_j}{\Pi(x)}(X_i^\mu-X_0^\mu)(X_j^\nu-X_0^\nu)\bar{e}_\mu e_\nu - (i\leftrightarrow j)\,, \tag{3.12}$$

(the X's here denote the same variables introduced in (3.6)), the second term of (2.30) can be represented by the form

$$\begin{aligned} \phi_{\text{II}}(x) &\equiv -\sum_{ij=1}^{\kappa}\left[\left(v^\dagger(x)C\right)_i\left(C^\dagger v(x)\right)_j - \left(v^\dagger(x)C\right)_j\left(C^\dagger v(x)\right)_i\right]\mathcal{A}_{ij} \\ &= -\frac{2}{S\,\Pi(x)}\frac{\sigma_a}{2i}\left[\sum_{ijmn=1}^{\kappa}(X_0^\mu X_i^\nu + X_i^\mu X_j^\nu + X_j^\mu X_0^\nu)\bar{\eta}_{\mu\nu a}\eta_{\lambda\delta b}\tilde{f}_{ij,mn}\left(z_0^\lambda z_m^\delta + z_m^\lambda z_n^\delta + z_n^\lambda z_0^\delta\right)\right]\phi_0^b. \end{aligned} \tag{3.13}$$

Based on (3.6) and (3.13), the full Higgs configuration in the JNR instanton background is

$$\begin{aligned} \phi(x) = -\frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}&\left[\sum_{ll'=0}^{\kappa}\rho_l^2\rho_{l'}^2X_l^\mu X_{l'}^\nu\bar{\eta}_{\mu\lambda a}\eta_{\nu\lambda b}\right. \\ &\left. +2\sum_{ijmn=1}^{\kappa}(X_0^\mu X_i^\nu + X_i^\mu X_j^\nu + X_j^\mu X_0^\nu)\bar{\eta}_{\mu\nu a}\eta_{\lambda\delta b}\tilde{f}_{ij,mn}\left(z_0^\lambda z_m^\delta + z_m^\lambda z_n^\delta + z_n^\lambda z_0^\delta\right)\right]\phi_0^b \end{aligned} \tag{3.14}$$

with the constants $\tilde{f}_{ij,mn}$ $(= -\tilde{f}_{ji,mn} = -\tilde{f}_{ij,nm})$ determined by inverting the linear inhomogeneous equations (3.9).

Based on (3.14), we will now produce explicit Higgs configurations appropriate to dyonic instantons with some small instanton number κ. In the $\kappa = 1$ JNR instanton background the corresponding Higgs configuration is very simple, being given by

$$\phi(x) = -\frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}\left[\bar{\eta}_{\mu\lambda a}\eta_{\nu\lambda b}\left(\rho_0^2 X_0 + \rho_1^2 X_1\right)^\mu \left(\rho_0^2 X_0 + \rho_1^2 X_1\right)^\nu\right]\phi_0^b \tag{3.15}$$

together with obvious identifications for S and $\Pi(x)$, i.e., $S = \rho_0^2 + \rho_1^2$ and $\Pi(x) = \frac{\rho_0^2}{|x-z_0|^2} + \frac{\rho_1^2}{|x-z_1|^2}$. It should be noted, however, that all $\kappa = 1$ dyonic instanton solutions are gauge-equivalent to corresponding 't Hooft-type dyonic instanton solutions. With $\kappa = 2$, nonzero elements of the matrix $\tilde{f}^{-1}$ — the matrix in terms of which the left hand side of (3.9) can be written $-(\tilde{f}^{-1})_{mn,rs}\tilde{\mathcal{A}}_{rs}$ (with $(\tilde{f}^{-1})_{mn,rs}\tilde{f}_{rs,ij} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}$) — are restricted to

$$\left(\tilde{f}^{-1}\right)_{12,12} = -\left(\tilde{f}^{-1}\right)_{21,12} = -\left(\tilde{f}^{-1}\right)_{12,21} = \left(\tilde{f}^{-1}\right)_{21,21} = -\left(\tilde{z}_{01}^2 + \tilde{z}_{12}^2 + \tilde{z}_{20}^2\right)\,, \tag{3.16}$$

and therefore we find, for nonzero elements of the matrix $\tilde{f}$,

$$\tilde{f}_{12,12} = -\tilde{f}_{21,12} = -\tilde{f}_{12,21} = \tilde{f}_{21,21} = -\frac{1}{2(\tilde{z}_{01}^2 + \tilde{z}_{12}^2 + \tilde{z}_{20}^2)}\,. \tag{3.17}$$

Using this result in (3.14), we obtain the appropriate Higgs configuration

$$\begin{aligned}\phi(x) = -\frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}\Bigg[&\bar{\eta}_{\mu\lambda a}\eta_{\nu\lambda b}(\rho_0^2 X_0 + \rho_1^2 X_1 + \rho_2^2 X_2)^\mu(\rho_0^2 X_0 + \rho_1^2 X_1 + \rho_2^2 X_2)^\nu \\ &-4\,\bar{\eta}_{\mu\nu a}\eta_{\lambda\delta b}(X_0^\mu X_1^\nu + X_1^\mu X_2^\nu + X_2^\mu X_0^\nu)\frac{(z_0^\lambda z_1^\delta + z_1^\lambda z_2^\delta + z_2^\lambda z_0^\delta)}{\tilde{z}_{01}^2 + \tilde{z}_{12}^2 + \tilde{z}_{20}^2}\Bigg]\phi_0^b\,.\end{aligned} \tag{3.18}$$

The result equivalent to this expression was obtained earlier in ref. [7]. Notice that this form exhibits a full symmetry under the interchange of three sets of JNR parameters, (ρ_0, z_0^μ), (ρ_1, z_1^μ) and (ρ_2, z_2^μ).

For $\kappa = 3$ it is convenient to define $\tilde{\mathcal{A}}_i$ $(i = 1, 2, 3)$ by $\tilde{\mathcal{A}}_{rs} = \epsilon_{rsi}\tilde{\mathcal{A}}_i$, so that (3.9) may be recast as

$$\begin{pmatrix} \tilde{z}_{23}^2 + \tilde{z}_{20}^2 + \tilde{z}_{30}^2 & -\tilde{z}_{30}^2 & -\tilde{z}_{20}^2 \\ -\tilde{z}_{30}^2 & \tilde{z}_{31}^2 + \tilde{z}_{30}^2 + \tilde{z}_{10}^2 & -\tilde{z}_{10}^2 \\ -\tilde{z}_{20}^2 & -\tilde{z}_{10}^2 & \tilde{z}_{12}^2 + \tilde{z}_{10}^2 + \tilde{z}_{20}^2 \end{pmatrix}\begin{pmatrix}\tilde{\mathcal{A}}_1 \\ \tilde{\mathcal{A}}_2 \\ \tilde{\mathcal{A}}_3\end{pmatrix} = \frac{1}{S}\eta_{\lambda\delta b}\phi_0^b\begin{pmatrix} z_0^\lambda z_2^\delta + z_2^\lambda z_3^\delta + z_3^\lambda z_0^\delta \\ z_0^\lambda z_3^\delta + z_3^\lambda z_1^\delta + z_1^\lambda z_0^\delta \\ z_0^\lambda z_1^\delta + z_1^\lambda z_2^\delta + z_2^\lambda z_0^\delta \end{pmatrix}. \tag{3.19}$$

After somewhat tedious algebra, one can determine $\tilde{\mathcal{A}}_i$ and hence the expression for $\phi_{\mathrm{II}}(x)$ (see (3.13)) also. With some rearrangements one can then write the explicit Higgs configuration that go with $\kappa = 3$ JNR instanton in the form

$$\begin{aligned}\phi(x) = -\frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}\Bigg[&\bar{\eta}_{\mu\lambda a}\eta_{\nu\lambda b}\left(\sum_{l=0}^{3}\rho_l^2 X_l^\mu\right)\left(\sum_{l'=0}^{3}\rho_{l'}^2 X_{l'}^\nu\right) \\ &-\frac{2}{D^{(3)}}\bar{\eta}_{\mu\nu a}\eta_{\lambda\delta b}\left\{\sum_{ll'=0}^{3} X_l^\mu X_{l'}^\nu\left[P_{ll'}^{(3)} z_l^\lambda z_{l'}^\delta + 2\sum_{k\neq(l,l')}^{3} P_{ll'k}^{(3)} z_{l'}^\lambda z_k^\delta + \sum_{kk'\neq(l,l')}^{3} P_{ll',kk'}^{(3)} z_k^\lambda z_{k'}^\delta\right]\right\}\Bigg]\phi_0^b\,,\end{aligned} \tag{3.20}$$

where $D^{(3)}$, the determinant of the 3×3 matrix appearing in the left hand side of (3.19), is given by

$$
\begin{aligned}
D^{(3)} = \; & \tilde{z}_{12}^2 \tilde{z}_{20}^2 \tilde{z}_{10}^2 + \tilde{z}_{31}^2 \tilde{z}_{30}^2 \tilde{z}_{10}^2 + \tilde{z}_{23}^2 \tilde{z}_{30}^2 \tilde{z}_{20}^2 + \tilde{z}_{12}^2 \tilde{z}_{23}^2 \tilde{z}_{31}^2 \\
& + \tilde{z}_{31}^2 \tilde{z}_{23}^2 \tilde{z}_{20}^2 + \tilde{z}_{12}^2 \tilde{z}_{23}^2 \tilde{z}_{30}^2 + \tilde{z}_{23}^2 \tilde{z}_{31}^2 \tilde{z}_{10}^2 + \tilde{z}_{12}^2 \tilde{z}_{31}^2 \tilde{z}_{30}^2 \\
& + \tilde{z}_{23}^2 \tilde{z}_{30}^2 \tilde{z}_{10}^2 + \tilde{z}_{31}^2 \tilde{z}_{30}^2 \tilde{z}_{20}^2 + \tilde{z}_{10}^2 \tilde{z}_{12}^2 \tilde{z}_{23}^2 + \tilde{z}_{31}^2 \tilde{z}_{12}^2 \tilde{z}_{20}^2 \\
& + \tilde{z}_{10}^2 \tilde{z}_{20}^2 \tilde{z}_{23}^2 + \tilde{z}_{12}^2 \tilde{z}_{20}^2 \tilde{z}_{30}^2 + \tilde{z}_{31}^2 \tilde{z}_{10}^2 \tilde{z}_{20}^2 + \tilde{z}_{12}^2 \tilde{z}_{10}^2 \tilde{z}_{30}^2 \, ,
\end{aligned}
\tag{3.21}
$$

and $P_{ll'}^{(3)}$, $P_{ll'k}^{(3)}$, $P_{ll',kk'}^{(3)}$ (with any of the indices l, l', k and k' taking values among 0, 1, 2, 3) are some quadratic polynomials of the $\tilde{z}^2$'s precise form of which we will specify below.

Observe that the determinant $D^{(3)}$ is a cubic polynomial of the six elements $\tilde{z}_{01}^2$, $\tilde{z}_{02}^2$, $\tilde{z}_{03}^2$, $\tilde{z}_{12}^2$, $\tilde{z}_{13}^2$ and $\tilde{z}_{23}^2$, satisfying the conditions that (i) no element appears more than once in each monomial, (ii) no given index $l(= 0, 1, 2, 3)$ appears more than twice in each monomial, and (iii) the full sum exhibits symmetry under the interchange of four JNR parameters (ρ_0, z_0^μ), (ρ_1, z_1^μ), (ρ_2, z_2^μ) and (ρ_3, z_3^μ). Then the polynomials $P_{ll'}^{(3)}$, each given by the sum of eight quadratic monomials of the $\tilde{z}^2$'s, are related to $D^{(3)}$ by

$$
P_{ll'}^{(3)} = \frac{\partial D^{(3)}}{\partial \tilde{z}_{ll'}^2} \quad \left(= P_{l'l}^{(3)} \right) \; . \tag{3.22}
$$

Now let $P_{ll'}^{(3)} \cap P_{kk'}^{(3)}$ denote the sum of all monomials which make simultaneous appearance in both polynomials $P_{ll'}^{(3)}$ and $P_{kk'}^{(3)}$. We then find that the polynomials $P_{ll'k}^{(3)}$ above, each corresponding to the sum of four quadratic monomials of the $\tilde{z}^2$'s, can be identified with

$$
P_{ll'k}^{(3)} = P_{ll'}^{(3)} \cap P_{l'k}^{(3)} \, , \tag{3.23}
$$

while the polynomials $P_{ll',kk'}^{(3)}$, in the last piece of (3.20), equal

$$
P_{ll',kk'}^{(3)} = \tilde{z}_{lk}^2 \tilde{z}_{l'k'}^2 \frac{\partial}{\partial \tilde{z}_{lk}^2} \frac{\partial}{\partial \tilde{z}_{l'k'}^2} \left(P_{ll'}^{(3)} \cap P_{kk'}^{(3)} \right) \; . \tag{3.24}
$$

Explicitly, we have

$$
\begin{aligned}
P_{01}^{(3)} &= \tilde{z}_{03}^2 \, \tilde{z}_{12}^2 + \tilde{z}_{12}^2 \, \tilde{z}_{23}^2 + \tilde{z}_{32}^2 \, \tilde{z}_{20}^2 + \tilde{z}_{02}^2 \, \tilde{z}_{21}^2 + \tilde{z}_{02}^2 \, \tilde{z}_{13}^2 + \tilde{z}_{23}^2 \, \tilde{z}_{30}^2 + \tilde{z}_{03}^2 \, \tilde{z}_{31}^2 + \tilde{z}_{13}^2 \, \tilde{z}_{32}^2 \, , \\
P_{12}^{(3)} &= \tilde{z}_{10}^2 \, \tilde{z}_{23}^2 + \tilde{z}_{13}^2 \, \tilde{z}_{32}^2 + \tilde{z}_{23}^2 \, \tilde{z}_{30}^2 + \tilde{z}_{03}^2 \, \tilde{z}_{31}^2 + \tilde{z}_{13}^2 \, \tilde{z}_{20}^2 + \tilde{z}_{10}^2 \, \tilde{z}_{03}^2 + \tilde{z}_{30}^2 \, \tilde{z}_{02}^2 + \tilde{z}_{20}^2 \, \tilde{z}_{01}^2 \, , \\
P_{012}^{(3)} &= P_{01}^{(3)} \cap P_{12}^{(3)} = \tilde{z}_{13}^2 \, \tilde{z}_{20}^2 + \tilde{z}_{23}^2 \, \tilde{z}_{30}^2 + \tilde{z}_{03}^2 \, \tilde{z}_{31}^2 + \tilde{z}_{13}^2 \, \tilde{z}_{32}^2 \, , \\
P_{0123}^{(3)} &= \tilde{z}_{02}^2 \tilde{z}_{13}^2 \, ,
\end{aligned}
\tag{3.25}
$$

etc. We remark that the resulting Higgs configuration has full symmetry under the exchange of four sets of JNR parameters.

Beyond $\kappa = 3$ the algebra involved in solving (3.9) becomes very complicated. But, inferring from the detailed analysis we performed for the case of $\kappa = 4$, the Higgs configuration for $\kappa \geq 4$ appears to be described by a direct extension of our $\kappa = 3$ formula (3.20), i.e., by using now the related determinant $D^{(\kappa)}$ (associated with the $\frac{\kappa(\kappa-1)}{2} \times \frac{\kappa(\kappa-1)}{2}$ matrix, formed from the coefficients multiplying the $\tilde{\mathcal{A}}$'s in (3.9)) and the $\tilde{z}^2$-dependent quantities $P^{(\kappa)}_{ll'}$, $P^{(\kappa)}_{ll'k}$ and $P^{(\kappa)}_{ll',kk'}$ given by

$$
\begin{aligned}
P^{(\kappa)}_{ll'} &= \frac{\partial D^{(\kappa)}}{\partial \tilde{z}^2_{ll'}}\,, \qquad\qquad P^{(\kappa)}_{ll'k} = P^{(\kappa)}_{ll'} \cap P^{(\kappa)}_{l'k}\,, \\
P^{(\kappa)}_{ll',kk'} &= \tilde{z}^2_{lk}\tilde{z}^2_{l'k'}\frac{\partial}{\partial \tilde{z}^2_{lk}}\frac{\partial}{\partial \tilde{z}^2_{l'k'}}\left(P^{(\kappa)}_{ll'} \cap P^{(\kappa)}_{kk'}\right)\,. \qquad (3.26)
\end{aligned}
$$

Actually, even for $\kappa = 2$, the corresponding form with $D^{(2)} = \tilde{z}^2_{01} + \tilde{z}^2_{12} + \tilde{z}^2_{20}$, $P^{(2)}_{ll'} = 1$ for $l \neq l'$, $P^{(2)}_{ll'k} = 1$ for $k \neq (l, l')$, and $P^{(2)}_{ll',kk'} \equiv 0$ reproduces the expression (3.18) exactly. For $\kappa = 4$, $D^{(4)}$ — a sixth-order polynomial satisfying the restrictions that (i) no element appears more than once in each monomial, (ii) no given index $l(= 0, 1, 2, 3, 4)$ appears more than three times in each monomial, (iii) no monomial of the type $\tilde{z}^2_{02}\tilde{z}^2_{03}\tilde{z}^2_{04}\tilde{z}^2_{12}\tilde{z}^2_{13}\tilde{z}^2_{14}$ is kept, and (iv) the full sum exhibits symmetry under the interchange of five sets of JNR parameters. In this case we verified that the full Higgs configuration is represented by a direct generalization of (3.20); no additional term is needed, whatsoever. We conjecture that this be the case for $\kappa \geq 5$ also. Accepting this, the full Higgs configurations appropriate to JNR dyonic instantons follow only if the suitable expression for the determinant associated with our linear equations (3.9) has been evaluated. The symmetry of the configuration under the exchange of JNR parameters will be automatic.

The electric charge for JNR dyonic instantons can be computed using our formula (2.40). Especially, for the $\kappa = 1, 2$ and 3 cases, one obtains following values:

$$
\kappa = 1: \quad Q_e = 2\pi^2 \mathrm{v}\frac{\rho_0^2\rho_1^2}{(\rho_0^2+\rho_1^2)^2}(z_1 - z_0)^2\,, \qquad (3.27)
$$

$$
\begin{aligned}
\kappa = 2: \quad Q_e = \frac{2\pi^2}{\mathrm{v}(\rho_0^2+\rho_1^2+\rho_2^2)^2}\Bigg\{ &\mathrm{v}^2\left[\rho_0^2\rho_1^2(z_0-z_1)^2+\rho_1^2\rho_2^2(z_1-z_2)^2+\rho_2^2\rho_0^2(z_2-z_0)^2\right] \\
&- \frac{4\left[\eta_{\mu\nu a}\phi_0^a\left(z_0^\mu z_1^\nu + z_1^\mu z_2^\nu + z_2^\mu z_0^\nu\right)\right]^2}{\tilde{z}^2_{01}+\tilde{z}^2_{12}+\tilde{z}^2_{20}}\Bigg\}\,, \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
\kappa = 3: \quad Q_e = \frac{2\pi^2}{\mathrm{v}\left(\sum_{l=0}^3 \rho_l^2\right)^2}\Bigg\{ &\mathrm{v}^2\left[\rho_0^2\rho_1^2(z_0-z_1)^2+\rho_0^2\rho_2^2(z_0-z_2)^2+\rho_0^2\rho_3^2(z_0-z_3)^2\right. \\
&\left.+\rho_1^2\rho_2^2(z_1-z_2)^2+\rho_1^2\rho_3^2(z_1-z_3)^2+\rho_2^2\rho_3^2(z_2-z_3)^2\right] \\
&-\frac{4P^{(3)}_{12}}{D^{(3)}}\left[\eta_{\mu\nu a}\phi_0^a(z_0^\mu z_1^\nu+z_1^\mu z_2^\nu+z_2^\mu z_0^\nu)\right]^2-\frac{4P^{(3)}_{23}}{D^{(3)}}\left[\eta_{\mu\nu a}\phi_0^a(z_0^\mu z_2^\nu+z_2^\mu z_3^\nu+z_3^\mu z_0^\nu)\right]^2 \\
&-\frac{4P^{(3)}_{31}}{D^{(3)}}\left[\eta_{\mu\nu a}\phi_0^a(z_0^\mu z_3^\nu+z_3^\mu z_1^\nu+z_1^\mu z_0^\nu)\right]^2 \\
&-\frac{8P^{(3)}_{123}}{D^{(3)}}\left[\eta_{\mu\nu a}\phi_0^a(z_0^\mu z_1^\nu+z_1^\mu z_2^\nu+z_2^\mu z_0^\nu)\right]\left[\eta_{\lambda\delta b}\phi_0^b\left(z_0^\lambda z_2^\delta+z_2^\lambda z_3^\delta+z_3^\lambda z_0^\delta\right)\right] \\
&-\frac{8P^{(3)}_{213}}{D^{(3)}}\left[\eta_{\mu\nu a}\phi_0^a(z_0^\mu z_1^\nu+z_1^\mu z_2^\nu+z_2^\mu z_0^\nu)\right]\left[\eta_{\lambda\delta b}\phi_0^b\left(z_0^\lambda z_3^\delta+z_3^\lambda z_1^\delta+z_1^\lambda z_0^\delta\right)\right]
\end{aligned}
$$

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$$-\frac{8P^{(3)}_{231}}{D^{(3)}}[\eta_{\mu\nu a}\phi_0^a(z_0^\mu z_2^\nu + z_2^\mu z_3^\nu + z_3^\mu z_0^\nu)]\Big[\eta_{\lambda\delta b}\phi_0^b\Big(z_0^\lambda z_3^\delta + z_3^\lambda z_1^\delta + z_1^\lambda z_0^\delta\Big)\Big]\Bigg\}. \tag{3.29}$$

The result (3.27) for $\kappa = 1$ JNR dyonic instanton is easily understood, based on the facts that (i) the configuration with JNR parameters (ρ_0, z_0^μ) and (ρ_1, z_1^μ) is gauge-equivalent to the 't Hooft-type dyonic instanton with size $\rho = \frac{\rho_0\rho_1}{\rho_0^2+\rho_1^2}|z_1 - z_0|$ and position $z^\mu = \frac{\rho_1^2 z_0^\mu + \rho_0^2 z_1^\mu}{\rho_0^2+\rho_1^2}$, and (ii) for the latter we obtained the result $Q_e = 2\pi^2 \mathrm{v}\, \rho^2$ already. Our expression (3.28) coincides with the result obtained earlier [7]. As regars our result (3.29) giving the value for $\kappa = 3$, we have checked explicitly that the given expression exhibits the full symmetry under the interchange of related four JNR parameters, despite its partly asymmetric appearance. We also remark that both the results (3.28) and (3.29) reduce to the corresponding electric charge values of 't Hooft-type dyonic instantons if we take the limit $\rho_0^2 \to \infty$, $z_0^2 \to \infty$ with the ratio ρ_0^2/z_0^2 held to 1.

4. Higgs zero locus and connection to supertubes

In the previous section, we obtained a very explicit form of Higgs configuration (3.20) for JNR-type dyonic instanton with $\kappa = 3$, and reported an observation that the same form for the Higgs solutions also goes for the $\kappa = 4$ case. Based on this observation, we propose the following form of Higgs configuration for an arbitrary topological charge κ:

$$\begin{aligned}\phi(x) = &-\frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}\Bigg[\bar\eta_{\mu\lambda a}\eta_{\nu\lambda b}\Big(\sum_{l=0}^{\kappa}\rho_l^2 X_l^\mu\Big)\Big(\sum_{l'=0}^{\kappa}\rho_{l'}^2 X_{l'}^\nu\Big)\\ &-\frac{2}{D^{(\kappa)}}\bar\eta_{\mu\nu a}\eta_{\lambda\delta b}\Bigg\{\sum_{ll'=0}^{\kappa} X_l^\mu X_{l'}^\nu\Big[P^{(\kappa)}_{ll'} z_l^\lambda z_{l'}^\delta + 2\sum_{k\neq(l,l')}^{\kappa} P^{(\kappa)}_{ll'k} z_{l'}^\lambda z_k^\delta + \sum_{kk'\neq(l,l')}^{\kappa} P^{(\kappa)}_{ll',kk'} z_k^\lambda z_{k'}^\delta\Big]\Bigg\}\Bigg]\phi_0^b\,.\end{aligned} \tag{4.1}$$

With this expression, we here present some analysis on the zeroes of the above Higgs field $\phi(x)$. For this purpose it is convenient to rewrite (4.1) as follows:

$$\begin{aligned}\phi(x) &\equiv \frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}M^{ab}\phi_0^b\\ &= \frac{1}{S\,\Pi(x)}\frac{\sigma_a}{2i}\Bigg[2\mathcal{F}^a\mathcal{F}^b - (\mathcal{F}^c\mathcal{F}^c - \mathcal{F}^0\mathcal{F}^0)\delta^{ab} + 2\epsilon^{abc}\mathcal{F}^c\mathcal{F}^0 + \sum_{ll'=0}^{\kappa}(V_{ll'})_a(W_{ll'})_b\Bigg]\phi_0^b\,,\end{aligned} \tag{4.2}$$

where we have defined

$$\mathcal{F}^\mu = \sum_{l=0}^{\kappa}\mathcal{F}_l^\mu\ , \qquad \mathcal{F}_l^\mu = \rho_l^2 X_l^\mu = \frac{\rho_l^2(x-z_l)^\mu}{|x-z_l|^2}\,, \tag{4.3}$$

$$(V_{ll'})_a = 2\bar\eta_{\mu\nu a}X_l^\mu X_{l'}^\nu\,, \tag{4.4}$$

$$(W_{ll'})_b = \frac{\eta_{\lambda\delta b}}{D^{(\kappa)}}\Bigg[P^{(\kappa)}_{ll'} z_l^\lambda z_{l'}^\delta + 2\sum_{k\neq(l,l')}^{\kappa} P^{(\kappa)}_{ll'k} z_{l'}^\lambda z_k^\delta + \sum_{kk'\neq(l,l')}^{\kappa} P^{(\kappa)}_{ll',kk'} z_k^\lambda z_{k'}^\delta\Bigg]\ . \tag{4.5}$$

Note that the Higgs field is linearly proportional to its asymptotic value ϕ_0^a. Thus the zero locus will be independent of the magnitude of the asymptotic value. As we increase its value, the corresponding electric charge also increases linearly, counterbalancing the force which is trying to shrink instantons. However, the zero locus will depend crucially on the orientation of the asymptotic value ϕ_0^a. Usually we fix this quantity to be diagonal and change the orientation of the instanton configuration, but here we may fix the instanton field configuration and consider changing the orientation of the asymptotic value for convenience. For Higgs fields to vanish, the matrix $M = M^{ab}$ (defined in (4.2)) should have an eigenvector with zero eigenvalue or $\det M = 0$. As it is a single equation on four coordinates x^μ, the equation $\det M = 0$ defines a 3-dimensional hypersurface. This hypersurface represents the collection of all zeroes of the Higgs field as we change the orientation of its asymptotic value. There are two parameters in choosing the orientation of the asymptotic Higgs field, implying that there is 1-dimensional zero locus for a given orientation as expected.

We shall begin our analysis of the Higgs zeroes with more in-depth study of the $\kappa = 2$ case, as ref. [7] on this case has studied only simpler specific cases. Since three position parameters z_l's can always be on a plane, we may take all z_l's to be points on the 1-2 plane. Let us restrict our attention to zeroes of Higgs appearing on $\mathbb{R}^3$ (corresponding to the choice $x^0 = 0$). Then all X_l^0's and other vectors related to the x^0-direction vanish, so that we can express the determinant of M by a simple form

$$\det M = |\vec{\mathcal{F}}|^2 \left(|\vec{\mathcal{F}}|^4 - \sum_{ll'} (V_{ll'})_3 (W_{ll'})_3 |\vec{\mathcal{F}}|^2 + 2 \sum_{ll'} (\vec{\mathcal{F}} \cdot \vec{V}_{ll'}) \mathcal{F}^3 (W_{ll'})_3 \right) , \qquad (4.6)$$

where the vector symbol means a vector in $\mathbb{R}^3$. Thus the existence of zeroes requires that the right hand side of (4.6) should vanish. There are two possibilities, and one is simply the zero total "force" condition $\vec{\mathcal{F}} = 0$. (In the expression (4.3), $\mathcal{F}_l^\mu$ has the same form as 2-dimensional Coulomb force due to a source at z_l with charge ρ_l^2, and so we will call them "forces" and $\mathcal{F}$ a total force). This gives rise to zeroes which correspond generically to two distinct points on the 1-2 plane, since there are three sources for the force. The other possibility is that the quantity inside the parentheses in (4.6) may vanish: this case is more complicated, and in fact contains the zeroes of the first possibility. Thus we examined the second possibility to obtain surfaces, on which the Higgs zeroes can lie, in figure 1 and figure 2 with some choice of JNR parameters. The zero loci found in ref. [7] are identified with the sections of the surfaces in figures 1 and 2 appearing on the 1-2 plane ($x^3 = 0$). Each of the sections are drawn in figure 1-c) and figure 2-b), respectively. The isolated points at $x^3 = 0$ in both surfaces correspond to the case with asymptotic values (with fixed $\mathrm{v} = |\phi_0|$) $\vec{\phi}_0 = (\mathrm{v}\cos\alpha, \mathrm{v}\sin\alpha, 0)$ for some α, while the circle at $x^3 = 0$ in figure 1-c) and the closed loop at $x^3 = 0$ in figure 2-b) are the results with $\vec{\phi}_0 = (0, 0, \mathrm{v})$.

Now let us concentrate on the most symmetric case shown in figure 1, and consider more complicated situation than that of [7], i.e., when we have nontrivial (ϕ_0^1, ϕ_0^2) with nonzero value of ϕ_0^3. Let us first focus on the section of the zero surface appearing in the 1-3 plane ($x^2 = 0$) as shown in figure 1-d). By symmetry we here have $\mathcal{F}^2 = 0$ and $\sum_{ll'} (V_{ll'})_2 (W_{ll'})_3 = 0$, so that the expression of the Higgs field (and hence the matrix M

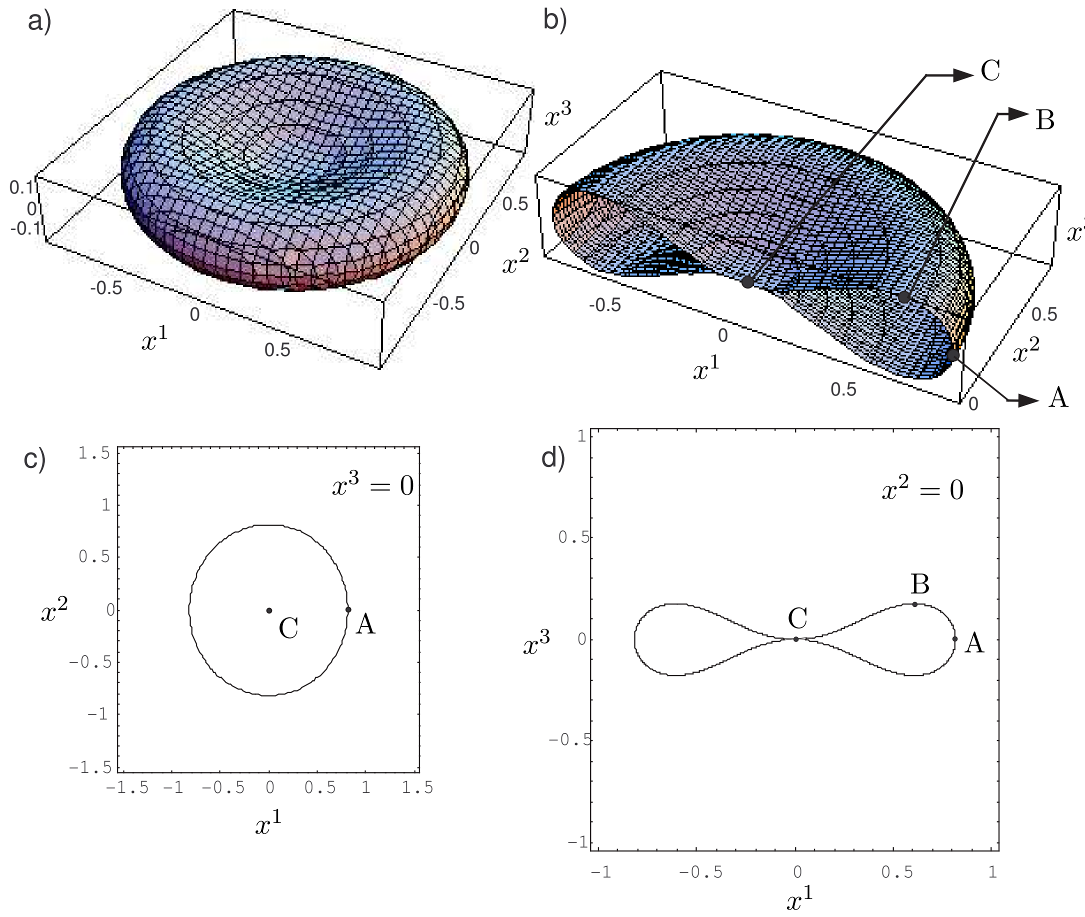


Figure 1: a) The surface of vanishing determinant of M at $x^0 = 0$, with identical size parameters $\rho_0 = \rho_1 = \rho_2 = 1$ and symmetric position parameters z_l's chosen at $z_0 = (-1, 0)$, $z_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $z_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. b) The half of the surface which shows the section $x^2 = 0$ of it. c) The section $x^3 = 0$ of the surface. d) The section $x^2 = 0$ of the surface.

also) takes a simpler form

$$\vec{\phi}(x) = \frac{1}{S\Pi(x)} \begin{pmatrix} (\mathcal{F}^1)^2 - (\mathcal{F}^3)^2 & 0 & 2\mathcal{F}^1\mathcal{F}^3 + \sum_{ll'}(V_{ll'})_1(W_{ll'})_3 \\ 0 & -((\mathcal{F}^1)^2 + (\mathcal{F}^3)^2) & 0 \\ 2\mathcal{F}^1\mathcal{F}^3 & 0 & -((\mathcal{F}^1)^2 - (\mathcal{F}^3)^2) + \sum_{ll'}(V_{ll'})_3(W_{ll'})_3 \end{pmatrix} \begin{pmatrix} \phi_0^1 \\ \phi_0^2 \\ \phi_0^3 \end{pmatrix}. \tag{4.7}$$

From the corresponding matrix M one can easily see that ϕ_0^2 should vanish and ϕ_0^1 is proportional to ϕ_0^3. If we start from $\vec{\phi}_0 = (0, 0, \mathrm{v})$ and perform a rotation in the 1-3 plane (say, to $\vec{\phi}_0 = (\mathrm{v}\sin\beta, 0, \mathrm{v}\cos\beta)$) in group space, we can see the zero locus lift away from the 1-2 plane. As we increase the angle β from zero to $\pi/2$, the part of the zero locus on the 1-3 plane moves along the curve depicted in figure 1-d) from the point A to the point C, via the point B. For this (most symmetric) case, simultaneous rotations of the 1-2 and

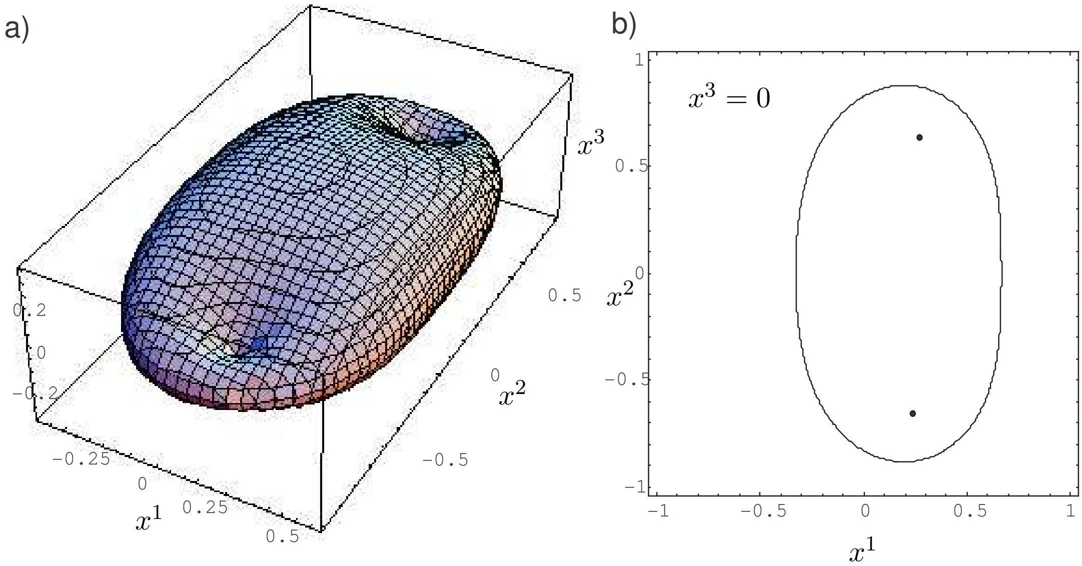


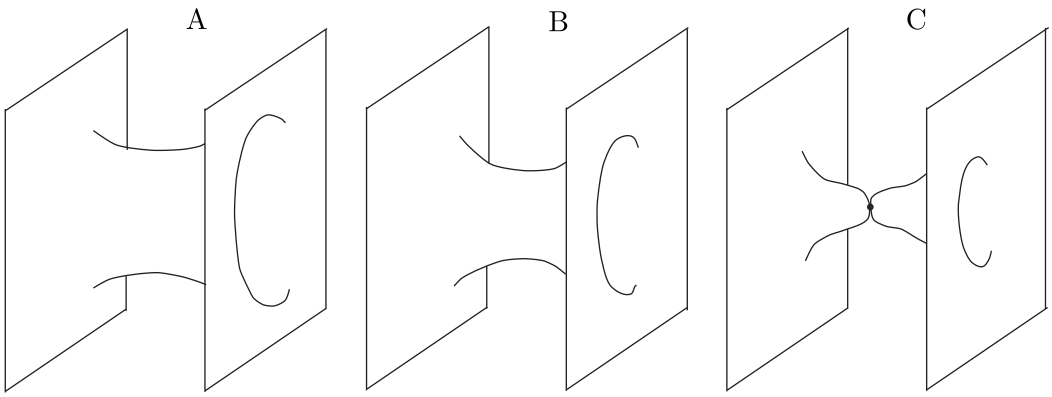
Figure 2: a) The surface $\det M = 0$ at $x^0 = 0$ with JNR parameters $\rho_0 = 2, \rho_1 = \rho_2 = 1, z_0 = (-1, 0), z_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $z_2 = (\frac{1}{2} - \frac{\sqrt{3}}{2})$. b) The section $x^3 = 0$ of the surface.

3-0 coordinates by a same amount of angle are symmetries of the given configuration. Thus the zero locus in this case (i.e., for a given β) will be a circle which remains invariant under these simultaneous rotations. This circle is a bit away from the 1-2 plane and also from the 3-0 plane,[2] and as one can see from the figure, it shrinks to the point C as one increases β to $\pi/2$. We can also turn on a nonzero ϕ_0^2 at this stage. By symmetry, the zero locus will again be a circle which remains invariant under the simultaneous rotations of the 1-2 and 3-0 coordinates by a same amount of angle. The intersecting point of this zero circle and the $x^0 = 0$ subsurface will be a generic point on the surface in figure 1-a).

As the eigenvalues of the asymptotic Higgs field can be interpreted as the asymptotic positions (multiplied by $1/i$) of two D4-branes, we fix the asymptotic magnitude v but change the gauge orientation of the Higgs field, which is equivalent to the changing of the gauge orientation of instantons while fixing the Higgs orientation. The deformation of the D4-branes indicated by the Higgs orientation corresponding to the points A, B and C in figure 1 are described by cartoon figures in figure 3. The cross section of the supertube collapses from a circle to a point as we change the orientation of the Higgs field.

For a more generic choice of JNR parameters, described in figure 2, we can again start from $\vec{\phi}_0 = (0, 0, \mathrm{v})$ and turn on nonzero ϕ_0^1, ϕ_0^2. As we noticed before, the zero locus starts from a closed loop and ends up with two points as we change the Higgs orientation from $(0, 0, \mathrm{v})$ to $(\mathrm{v}\cos\alpha, \mathrm{v}\sin\alpha, 0)$ (α is an arbitrary angle here) in this case. The zero locus will get separated from the 1-2 plane and we can read the shape of the zero locus from figure 2. As we know the development of the Higgs zero locus under rotations of the Higgs

[2]From the Higgs field configuration (4.2), we notice that the points in the subsurface $\phi_0^2 = 0$ of the 3-dimensional hypersurface $\det M = 0$ can lie outside the subsurface $x^0 = 0$ of $\det M = 0$.



$\vec{\phi}_0 = (0, 0, \mathrm{v})$ $\vec{\phi}_0 = (\mathrm{v}\sin\beta, 0, \mathrm{v}\cos\beta)$ $\vec{\phi}_0 = (\mathrm{v}, 0, 0)$

Figure 3: The geometries of D4-brane-supertube configurations corresponding to the cases described in figure 1. The cross sections of the supertubes are always circluar in these cases. If we fix the distance (i.e., set $|\phi_0| = \mathrm{v} = \text{const.}$) between two D4-branes, the change from A to C becomes a rotation in the group space. After this ninety degrees rotation, the supertube shrinks to a point.

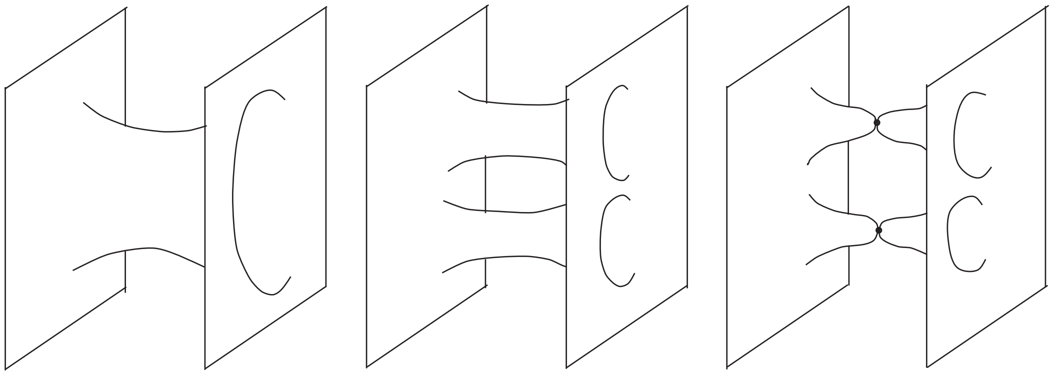


Figure 4: The geometries of D-brane configurations corresponding to the situations described in figure 2. In this figure, rotating ϕ_0, a supertube splits into two supertubes.

orientation, we can draw the cross section of the supertube connecting D4-branes as in figure 4, which shows the break-up of a single cross section to two cross sections.

Next we discuss the zero locus in the Higgs solutions for the $\kappa = 3$ case. Here, although we can generally put four position parameters z_l only on S^2, we put them on the 1-2 plane for simplicity. In this case the determinant of the matrix M is again given by our formula (4.6), from which we obtain a surface of the Higgs zeroes as shown in figure 5. As one can see from this figure, the zero locus of the Higgs field will change from a closed curve to three isolated points on the 1-2 plane, as we change the orientation of the Higgs field from $(0, 0, \mathrm{v})$ to $(\mathrm{v}\cos\alpha, \mathrm{v}\sin\alpha, 0)$. Related D4-brane-supertube configurations are given in figure 6 with three bridges between the two D4-branes.

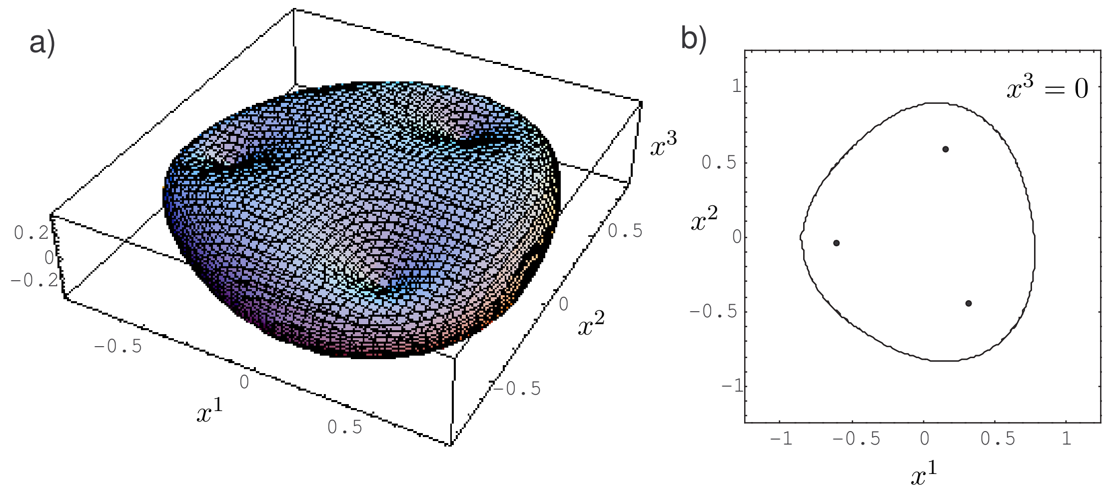


Figure 5: a) The surface $\det M = 0$ at $x^0 = 0$ with the instanton number $\kappa = 3$, $\rho_0 = 0.8, \rho_1 = 1, \rho_2 = 1.3, \rho_3 = 1.2, z_0 = (-0.9, 0), z_1 = (0, 1), z_2 = (1, 0)$ and $z_3 = (0, -1)$. In general, four position parameters z_l's can always be located on a sphere S^2. However, considering the facts that the JNR solutions are conformally covariant and a plane is conformally related to S^2, we can take the position parameters on a plane without an excessive simplification. b) The section $x^3 = 0$ of the surface.

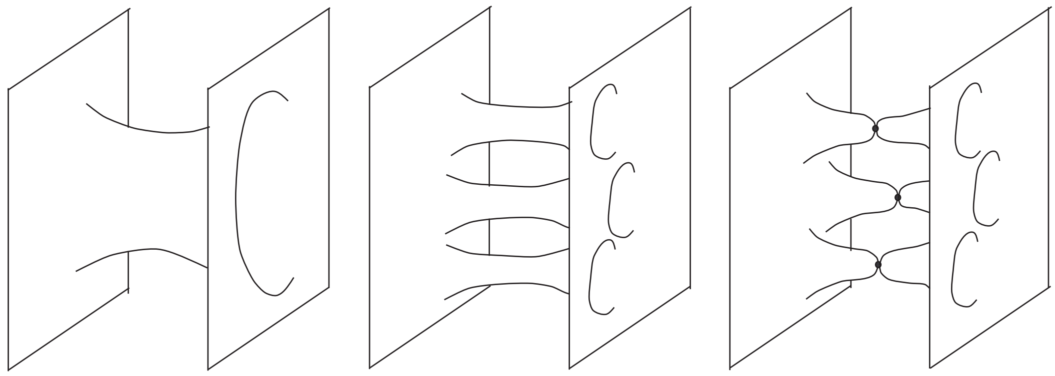


Figure 6: The D-brane realizations of the situations in figure 5. We expect that there happens splitting from one supertube to three supertubes.

From what we have found above, one may make a conjecture that the zero locus of the Higgs field for the instanton number κ with the SU(2) gauge group can have at most κ disconnected components. However, since we know neither the fully explicit κ-instanton solutions nor the Higgs field configurations representing dyonic instantons of the most general type, it remains to be seen.

The JNR instanton solution has a residual local gauge symmetry [14] that we have already mentioned above. For any value of κ, if all $\kappa+1$ JNR position parameters are on a circle or a line, there is a one-parameter family of JNR parameters which are related by local

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gauge transformations. Specifically, the $\kappa = 2$ case has been explored in detail in ref. [22], and this one-parameter family is called a porism. In [7], this porism structure was rederived using the fact that zeroes of the Higgs field is gauge invariant. Now we would like to understand the one-parameter gauge family of the $\kappa = 3$ JNR solutions in the same vein below.

For $\kappa = 3$, if we put all z_l's on the 1-2 plane and take $\phi_0^3 = 0$, then the second term of the Higgs field, i.e., $\phi_{\rm II}$ (see (3.13)) vanishes. So the condition for Higgs zeroes becomes just the zero "force" condition, $\vec{\mathcal{F}} = 0$. This zero "force" condition can be represented by a complex equation

$$\sum_{l=0}^{3} \frac{\rho_l^2}{(w - w_l)} = 0\,, \tag{4.8}$$

where $w \equiv x^1 + ix^2$ and w_l's denote the JNR position parameters on a circle in the complex plane. We will write $w_l = Re^{i\theta_l}$, $l = 0, 1, 2$ here. We can rearrange this condition, to write it by a complex cubic equation with three complex parameters

$$w^3 - RC_1 w^2 + R^2 C_2 w - R^3 C_3 = 0\,, \tag{4.9}$$

where

$$C_1 = \sum_{l=0}^{3} (1 - \lambda_l) e^{i\theta_l}\,, \tag{4.10a}$$

$$C_2 = e^{i(\theta_0 + \cdots + \theta_3)} \sum_{l=0}^{3} \sum_{k \neq l}^{3} \lambda_l e^{-i(\theta_l + \theta_k)}\,, \tag{4.10b}$$

$$C_3 = e^{i(\theta_0 + \cdots + \theta_3)} \sum_{l=0}^{3} \lambda_l e^{-i\theta_l}\,. \tag{4.10c}$$

Here we have used notations $\lambda_l = \rho_l^2/S$ ($S = \sum_{l=0}^{3} \rho_l^2$), thus $\sum_{l=0}^{3} \lambda_l = 1$. Now we would like to find a one-parameter family of the variation in the JNR size and position parameters which can be identified with a residual gauge transformation. As the Higgs field transforms homogeneously under local gauge transformations, zeroes of the Higgs field must be gauge invariant. Therefore, solutions of the complex equation (4.9) should not change if gauge transformations are performed. Then the C_i's ($i = 1, 2, 3$) must be invariant under gauge transformations. There are three complex conditions for this invariance, i.e., $\delta C_{1,2,3} = 0$ and seven parameters[3] which can be varied. We are thus left with one-parameter family of variation, related to the residual gauge transformation. If we consider $\delta \bar{C}_1 + e^{-i(\theta_0 + \cdots + \theta_3)} \delta C_3 = 0$, we get the condition

$$i \sum_l \left(\sum_k e^{-i\theta_k} \lambda_k - e^{-i\theta_l} \right) \delta\theta_l \equiv i \sum_l A_l \delta\theta_l = 0\,. \tag{4.11}$$

Considering the other equations $\delta C_2 = 0$ and $\delta C_3 = 0$, we can express $\delta\lambda_l$'s as linear combinations of $\delta\theta_l$. Due to the constraint $\sum_l \lambda_l = 1$, $\sum_l \delta\lambda_l$ should vanish, so the equations

[3]These are four λ_l's and four θ_l's with a constraint $\sum_{l=0}^{3} \lambda_l = 1$.

which we have to solve are one complex equation (4.11) and

$$\sum_l \delta\lambda_l \equiv \frac{\sum_l B_l \delta\theta_l}{\cos\left(\frac{\theta_0+\theta_1-\theta_2-\theta_3}{2}\right)+\cos\left(\frac{\theta_0+\theta_2-\theta_1-\theta_3}{2}\right)+\cos\left(\frac{\theta_0+\theta_3-\theta_1-\theta_2}{2}\right)} = 0 \ , \qquad (4.12)$$

where B_l is given by

$$\begin{aligned} B_l = \ & \frac{i}{2} e^{\frac{i}{2}(\theta_0+\cdots+\theta_3)} \Bigg[\sum_k \lambda_k e^{-2i\theta_k} - 2\lambda_l e^{-2i\theta_l} + e^{-i\theta_l} \left(\sum_k \lambda_k e^{-i\theta_k} \right) \\ & - \left(\sum_k e^{-i\theta_k} \right) \left(\sum_{l'} \lambda_{l'} e^{-i\theta_{l'}} - \lambda_l e^{-i\theta_l} \right) \Bigg] \\ & + \text{ complex conjugate.} \end{aligned} \qquad (4.13)$$

The equations (4.11) and (4.12) are solved by choosing the parametrization

$$\delta\theta_l = i \sum_{l_1,l_2,l_3=0}^{3} \varepsilon^{l\, l_1 l_2 l_3} A_{l_1} \bar{A}_{l_2} B_{l_3} \ \delta\tau \ . \qquad (4.14)$$

Thus we have obtained an one-parameter family of variations. We believe that this one-parameter family of variations corresponds to the residual gauge symmetry of JNR instantons; but, to be definite, some further check will be required.

This consideration is actually valid even for the general κ case. In the general case we have a κ-th order complex equation with κ complex parameters, and there are also $2\kappa+1$ JNR position and size parameters on a circle; therefore, we are left with an one-parameter family of gauge transformations.

5. Conclusion and discussions

In this work we have studied BPS dyonic instantons in the Coulomb phase, and presented a general formalism to find the Higgs field satisfying the covariant Laplace equation in the general ADHM framework for the instantons. Especially, we found the explicit expression for the Higgs field solution in the Jackiw-Nohl-Rebbi three- or four-instanton background. In addition we explored in detail the zero locus of the Higgs field for two (and three, partially) instanton case and studied some aspect of the residual gauge freedom for JNR three-instanton solutions.

Our analysis shows that the Higgs solution in the instanton background has a very rich structure. While our detailed study was restricted to the case with the Jackiw-Nohl-Rebbi three- of four-instanton backgrounds, the general three-instanton solution has been found in refs. [12, 23] and the structure of related Higgs solutions need to be analyzed also.

Our dyonic instantons have very large degeneracy even when the instanton number and the electric charge are fixed. The moduli space dynamics of dyonic instantons is a phase space dynamics in the sense that the first order in time derivative term dominates. (This can be seen easily from the moduli-space dynamics of instantons). As explained in ref. [1], the moduli space dynamics of the instanton is also corrected by a potential term given by the Killing vector related to the symmetry breaking. The electric charge interaction and the potential are thus balanced and for the resulting BPS configuration the relative motion moduli space dynamics becomes first order in time. The supersymmetric generalization and detailed exploration of this dynamics needs a further consideration.

In a similar way to supertubes, dyonic instantons also carry nonzero angular momentum in 4-dimensions. One may split the angular momentum value into the self-dual and anti-self-dual parts. The detailed evaluation of the angular momentum in terms of the ADHM data remains to be done. There is a close relation between the shape of the zero locus and the magnitude of the angular momentum. For a given instanton number and electric charge, one expect that there exists an upper bound on the angular momentum. It can easily be estimated by the supertube analysis done in ref. [8]. In the large instanton limit and a circular magnetic monopole string case, one can approximate the magnetic monopole string as a straight string locally. For a given asymptotic Higgs expectation value v, the tension of the string and the momentum density is fixed as $4\pi \mathrm{v}/e^2$. There is one parameter h which allows to write the instanton energy density and charge density as $4\pi h/e^2$ and $4\pi \mathrm{v}/(e^2 h)$, respectively, so that their product is independent of h. For a circle-shaped monopole string of radius R lying on the 1-2 plane, the total instanton number would be $\kappa = Rh$ so that its energy becomes $\mathcal{E} = 8\pi^2 Rh/e^2$, and the total electric charge would be $Q_e = 8\pi^2 \mathrm{v} R/(e^2 h)$. The estimated value of the maximal angular momentum would then be

$$J_{12} = \frac{8\pi^2}{e^2} \mathrm{v} R^2 = \kappa Q_e \,, \tag{5.1}$$

which is independent of R (as κ and Q_e are kept fixed) and so of the density. It would be nice to verify that the maximal value of the angular momentum is indeed of this form. Related to this, the angular momentum in the $\kappa = 2$ case was studied by the authors of refs. [9, 10] in a field theory and in a supergravity model respectively.

Acknowledgments

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A. Direct verification of our Higgs solution

In the main text our Higgs solution (2.30) in the ADHM background was obtained from the asymptotic limit of the related scalar propagator. We shall here provide a direct check on this result by showing that the expression (2.30) indeed solves the covariant Laplace equation (2.7). For this purpose it is convenient to write the result (2.30) in the form

$$\phi(x) = v^\dagger(x) W v(x) \,, \tag{A.1}$$

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introducing a x-independent $(\kappa+1)\times(\kappa+1)$ matrix of quaternions

$$W = V\phi_0 V^\dagger - 2C\mathcal{A}C^\dagger \ . \tag{A.2}$$

Also useful is the well-known result that, as we represent the ADHM constraint by $\Delta^\dagger(x)\Delta(x) = f^{-1}(x)e_0$ ($f(x)$ is a real, invertible $\kappa\times\kappa$ matrix), we have [11–13]

$$1 - v^\dagger(x)v(x) = \Delta(x)f(x)\Delta^\dagger(x) \ . \tag{A.3}$$

From (A.1) and (A.2), it is not difficult to derive

$$\begin{aligned} D_\mu\phi &= \partial_\mu v^\dagger \left(1 - vv^\dagger\right) Wv + v^\dagger W \left(1 - vv^\dagger\right)\partial_\mu v \\ &= -v^\dagger\left(\partial_\mu\Delta\right) f\Delta^\dagger W v - v^\dagger W\Delta f\left(\partial_\mu\Delta^\dagger\right) v \,, \end{aligned} \tag{A.4}$$

where we used the relation (A.3) as well as the equations in (2.9). Then taking the covariant derivative once more with the thus-obtained expression, we find, after somewhat lengthy algebra, the following expression:

$$D_\mu D_\mu\phi = -4v^\dagger\left\{CfC^\dagger, W\right\}v + 4v^\dagger Cf\cdot \mathrm{tr}\Delta^\dagger W\Delta\cdot fC^\dagger v \ . \tag{A.5}$$

To obtain this form, we made use of the fact that $e_\mu q\bar{e}_\mu = 2\mathrm{tr}\, q$ for any quaternion q.

We may now insert the expression (A.2) for W into the right hand side of (A.5). Then, due to the fact that $C^\dagger V = V^\dagger C = 0$ (see (2.22)), the first piece in (A.5) can be written as

$$-4v^\dagger\left\{CfC^\dagger, W\right\}v = 8v^\dagger C\left(fC^\dagger C\mathcal{A} + \mathcal{A}C^\dagger Cf\right)C^\dagger v \ . \tag{A.6}$$

Also, if the expression (A.2) is used in the tr-term from the second piece of (A.5), it can be reduced to

$$\begin{aligned} \mathrm{tr}\Delta^\dagger W\Delta \,=\, & \mathrm{tr}B^\dagger V\phi_0 V^\dagger B - 2C^\dagger C\mathcal{A}f^{-1} - 2f^{-1}\mathcal{A}C^\dagger C \\ & -\mathrm{tr}\left\{B^\dagger[C\mathcal{A}, C^\dagger]B + [B^\dagger, C^\dagger C\mathcal{A}]B + B^\dagger[C^\dagger, \mathcal{A}C^\dagger]B + B^\dagger[\mathcal{A}C^\dagger C, B]\right\} \ . \end{aligned} \tag{A.7}$$

To obtain this result, we made use of the observation that all quadratic and linear terms in x from the expression

$$\mathrm{tr}\left\{\Delta^\dagger[C\mathcal{A}, C^\dagger]\Delta + [\Delta^\dagger, C^\dagger C\mathcal{A}]\Delta + \Delta^\dagger[C, \mathcal{A}C^\dagger]\Delta + \Delta^\dagger[\mathcal{A}C^\dagger C, \Delta]\right\} \tag{A.8}$$

cancel, to leave only the x-independent contribution equal to the last tr-term in (A.7). As we use the results (A.6) and (A.7) in (A.5), we are then left with the expression

$$\begin{aligned} D_\mu D_\mu\phi = 4v^\dagger Cf\cdot\Big[&\mathrm{tr}B^\dagger V\phi_0 V^\dagger B - \mathrm{tr}\Big\{B^\dagger[C\mathcal{A}, C^\dagger]B + [B^\dagger, C^\dagger C\mathcal{A}]B \\ & + B^\dagger[C, \mathcal{A}C^\dagger]B + B^\dagger[\mathcal{A}C^\dagger C, B]\Big\}\Big]\cdot fC^\dagger v \ . \end{aligned} \tag{A.9}$$

If we here define a quantity R, an antisymmetric $\kappa\times\kappa$ matrix, by

$$R = \mathrm{tr}\left\{2B^\dagger C\mathcal{A}C^\dagger B - C^\dagger C\mathcal{A}B^\dagger B - B^\dagger B\mathcal{A}C^\dagger C\right\} \tag{A.10}$$

and a κ-column vector $\tilde{v} = 2fC^\dagger v$, (A.9) can be further simplified to

$$D_\mu D_\mu \phi = \tilde{v}^\dagger \cdot \left[\mathrm{tr} B^\dagger V \phi_0 V^\dagger B - R \right] \cdot \tilde{v} \ . \tag{A.11}$$

Based on (A.11), we conclude that the Higgs configuration (A.1) corresponds to the solution of (2.7) only if the matrix $\mathcal{A} = (\mathcal{A}_{ij})$ satisfies the linear inhomogeneous equations

$$\begin{aligned}
\mathrm{tr} \left(B^\dagger V \right)_m \phi_0 \left(V^\dagger B \right)_n \ &= \ R_{mn} \\
= \ \frac{1}{2} \mathrm{tr} &\left\{ 2B^\dagger C \mathcal{A} C^\dagger B - C^\dagger C \mathcal{A} B^\dagger B - B^\dagger B \mathcal{A} C^\dagger C \right\}_{mn} - (m \leftrightarrow n) \\
= \left[\frac{1}{2} \mathrm{tr} \left\{ 2 \left(C^\dagger B \right)_{mr} \left(B^\dagger C \right)_{sn} - \left(C^\dagger C \right)_{mr} \left(B^\dagger B \right)_{sn} \right. \right. &\left. \left. - \left(B^\dagger B \right)_{mr} \left(C^\dagger C \right)_{sn} \right\} - (m \leftrightarrow n) \right] \mathcal{A}_{rs} \ .
\end{aligned} \tag{A.12}$$

These coincide with the equations we found for $\mathcal{A}$ in the main text, (2.29). This completes the verification.

B. Computation of electric charge

We shall here present the derivation of our expression (2.40) for the electric charge of dyonic instantons. General SU(2) dyonic instantons are described by the ADHM instantons (see (2.8) – (2.10)) and the corresponding Higgs field, written conveniently in our form (A.1) with matrix W (see (A.2)). Then, thanks to the BPS equations (2.6), we can express the electric field E_μ as

$$\begin{aligned}
E_\mu \ &= \ D_\mu \phi \ = \ \partial_\mu \phi + [v^\dagger \partial_\mu v, \phi] \\
&= \ \left(\partial_\mu v^\dagger \right) W v + v^\dagger W \partial_\mu v - \left(\partial_\mu v^\dagger \right) v v^\dagger W v - v^\dagger W v v^\dagger \partial_\mu v \ .
\end{aligned} \tag{B.1}$$

Using (A.3), this can be rewritten in the form

$$\begin{aligned}
E_\mu \ &= \ -v^\dagger (\partial_\mu \Delta) f \Delta^\dagger W v - v^\dagger W \Delta f \left(\partial_\mu \Delta^\dagger \right) v \\
&= \ v^\dagger (C e_\mu) f \left(B^\dagger - \bar{x} C^\dagger \right) W v + v^\dagger W (B - Cx) f \left(\bar{e}_\mu C^\dagger \right) v \, ,
\end{aligned} \tag{B.2}$$

where we used also the relation $(\partial_\mu v^\dagger)\Delta = -v^\dagger \partial_\mu \Delta$ and $\Delta^\dagger \partial_\mu v = -(\partial_\mu \Delta^\dagger) v$ (following from (2.9)). Inserting (A.2) for W and using $C^\dagger V = V^\dagger C = 0$, it is possible to recast the above expression in the form

$$\begin{aligned}
E_\mu \ &= v^\dagger C e_\mu f B^\dagger V \phi_0 V^\dagger v + v^\dagger V \phi_0 V^\dagger B f \bar{e}_\mu C^\dagger v \\
&\quad -2 v^\dagger C e_\mu f (B^\dagger C - \bar{x} C^\dagger C) \mathcal{A} C^\dagger v - 2 v^\dagger C \mathcal{A} (C^\dagger B - C^\dagger C x) f \bar{e}_\mu C^\dagger v \, ,
\end{aligned} \tag{B.3}$$

which is rather complicated, but contains no derivative.

The electric charge (2.5) is given by the surface integral

$$Q_e = - \oint_{S^3_\infty} \mathrm{d}S_\mu \ \frac{1}{\mathrm{v}} \mathrm{tr}(\phi E_\mu) = - \lim_{|x| \to \infty} \int \mathrm{d}\Omega(\hat{x}) \ |x|^3 \hat{x}_\mu \ \frac{1}{\mathrm{v}} \mathrm{tr}(\phi E_\mu) \ . \tag{B.4}$$

Taking the asymptotic behaviors (2.20)–(2.23) and that of $f(x)$

$$f(x) = \frac{1}{|x|^2} f_0 + \mathcal{O}\left(\frac{1}{|x|^3}\right), \quad (f_0 \equiv (C^\dagger C)^{-1}) \tag{B.5}$$

(as follows from studying $\Delta^\dagger \Delta = (B^\dagger - \bar{x} C^\dagger)(B - Cx) = f^{-1}$) into account, we can get the asymptotic behavior of E_μ

$$\begin{aligned} E_\mu \sim & \frac{1}{|x|^3}\left[\bar{g} V^\dagger B \hat{\bar{x}} e_\mu f_0 B^\dagger V \phi_0 g + \bar{g}\phi_0 V^\dagger B f_0 \bar{e}_\mu \hat{x} B^\dagger V g\right] \\ & + \frac{1}{|x|^4}\left[2\bar{g} V^\dagger B \hat{\bar{x}} e_\mu f_0 (\bar{x} C^\dagger C - B^\dagger C)\mathcal{A}\hat{x} B^\dagger V g + 2\bar{g} V^\dagger B \hat{\bar{x}} \mathcal{A}(C^\dagger C x - C^\dagger B) f_0 \bar{e}_\mu \hat{x} B^\dagger V g\right]. \end{aligned} \tag{B.6}$$

Based on this result and (2.12), one finds the following expression for the integrand of (B.4):

$$\begin{aligned} |x|^3 \hat{x}_\mu \mathrm{tr}(\phi E_\mu) = & \; |x|^3 \hat{x}_\mu \mathrm{tr}\left\{(\bar{g}\phi_0 g)\cdot \frac{1}{|x|^3}\left[\bar{g} V^\dagger B \hat{\bar{x}} e_\mu f_0 B^\dagger V \phi_0 g + \bar{g}\phi_0 V^\dagger B f_0 \bar{e}_\mu \hat{x} B^\dagger V g \right.\right. \\ & \left.\left. + 2\bar{g} V^\dagger B \hat{\bar{x}} e_\mu f_0 \hat{\bar{x}} C^\dagger C \mathcal{A} \hat{x} B^\dagger V g + 2\bar{g} V^\dagger B \hat{\bar{x}} \mathcal{A} C^\dagger C \hat{x} f_0 \bar{e}_\mu \hat{x} B^\dagger V g\right] + \mathcal{O}\left(\frac{1}{|x|^4}\right)\right\}. \end{aligned} \tag{B.7}$$

Using the cyclic property of the trace, $\bar{g} g = 1$ and $\hat{\bar{x}}\hat{x} = 1$ (g and $\hat{x} \equiv \hat{x}_\mu e_\mu$ are unit quaternions), a very simple expression

$$\lim_{|x|\to\infty} |x|^3 \hat{x}_\mu \mathrm{tr}(\phi E_\mu) = \mathrm{tr}\left\{2\phi_0^2 V^\dagger B (C^\dagger C)^{-1} B^\dagger V + 4\phi_0 V^\dagger B \mathcal{A} B^\dagger V\right\} \tag{B.8}$$

can be obtained from (B.7). From this asymptotic form one can easily deduce that the electric charge is given by the form (2.40).

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